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Algebraic Reasoning over Relational Structures

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Abstract

Many important computational structures involve an intricate interplay between algebraic features (given by operations on the underlying set) and relational features (taking account of notions such as order or distance). This paper investigates algebras over relational structures axiomatized by an infinitary Horn theory, which subsume, for example, partial algebras, various incarnations of ordered algebras, quantitative algebras introduced by Mardare, Panangaden, and Plotkin, and their recent extension to generalized metric spaces and lifted algebraic signatures by Mio, Sarkis, and Vignudelli. To this end, we develop the notion of clustered equation, which is inspired by Mardare et al.'s basic conditional equations in the theory of quantitative algebras, at the level of generality of arbitrary relational structures, and we prove that it is equivalent to an abstract categorical form of equation earlier introduced by Milius and Urbat. Our main results are a family of Birkhoff-type variety theorems (classifying the expressive power of clustered equations) and an exactness theorem (classifying abstract equations by a congruence property).

Keywords: Relational Structure, Algebra, Variety, Birkhoff, Equation

1 Introduction

The axiomatization of data types by operations (constructors) and equations that these ought to satisfy is an important tool in algebraic specifications. Accordingly, the theory of models of equational specifications is a topic of great interest in both mathematics and computer science. One key result is Birkhoff's seminal variety theorem, also known as the HSP theorem [7]. It states that a class of algebras over a signature Σ is a *variety* (i.e. axiomatizable by equations $s = t$ between Σ-terms) iff it is closed under *h*omomorphic

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images, *s*ubalgebras, and *p*roducts. Birkhoff also introduced a complete deduction system for reasoning about equations.

In modern algebraic approaches to the semantics of programming languages, data types and computational effects, models often involve an intricate interplay between algebraic features given by operations on the underlying set and relational features taking account of notions of order or distance. For example, Bloom [8] introduced *ordered algebras* (posets equipped with monotone operations) and established a variety theorem for them along with a complete deduction system. Here, the role of equations $s = t$ is taken over by inequations $s \leq t$. Another example is that of *quantitative algebras* (metric spaces equipped with nonexpansive operations), introduced by Mardare, Panangaden, and Plotkin [15,16], which naturally arise as semantic domains in the theory of probabilistic computation. In the quantitative setting, equations $s =_\varepsilon t$ are parameterized by a non-negative real number ε and interpreted as "s and t have distance at most ε". Among the main results of the latter work are a variety theorem for quantitative algebras and a complete deduction system. However, the underlying notion of quantitative algebra has subsequently turned out to be too restrictive for some applications, e.g. in quantitative term rewriting [11] and machine learning [9]. Therefore, Mio, Sarkis, and Vignudelli [20,21] recently proposed a generalization of it in two directions: (1) metric spaces are relaxed to *generalized metric spaces* where only a fixed but arbitrary subset of the axioms of a metric is required to hold; and (2) nonexpansivity of operations $\sigma_A \colon A^n \to A$ is not required w.r.t. the usual product metric on A^n, but w.r.t. to an arbitrary functorial choice of a metric on A^n (which might depend on σ), specified by a lifting of the set functor $(-)^n$. In this setting they present a complete deduction system for quantitative equations. However, a variety theorem classifying the expressive power of quantitative equations over lifted signatures is still missing.

It is one of the goals and motivations of our paper to fill this gap. On the way, we will move beyond the metric setting and investigate algebras and equational theories over general relational structures axiomatized by an infinitary Horn theory. This not only highlights that the precise nature of the underlying structures is largely irrelevant from the perspective of algebraic reasoning, but also allows us to uniformly cover a number of additional settings of interest, including partial algebras, various types of ordered algebras, and quantitative algebras with quantities beyond non-negative real numbers.

The main new concept developed in our paper is that of a *c-clustered equation* (parametric in a cardinal number c) for relational algebras. In the special case of quantitative algebras over metric spaces, this notion has previously appeared in the work of Milius and Urbat [18], where it is introduced as a variant of Mardare et al.'s basic conditional equations [16]. Informally, c-clustered equations can express properties of algebras that involve conditions on their variables, e.g. a conditional commutative law $x =_{1/2} y \vdash x \bullet y =_{1/4} y \bullet x$ for quantitative algebras with a binary operation $\bullet$ or $x \leq y \vdash x \bullet y \leq y \bullet x$ for ordered algebras. The parameter c controls the level of connectedness between the variables appearing in the premise. Our main result is the variety theorem for c-varieties (Theorem 4.16), which states that a class of algebras for a (possibly infinitary) lifted signature is axiomatizable by c-clustered equations iff it is closed under *c-reflexive* quotients, subalgebras, and products. Note that, unlike in Birkhoff's classical variety theorem, c-varieties need not be closed under all quotients but only under those from a certain class of quotients depending on the parameter c.

Our approach to equations and varieties is based on category theory. Specifically, we make the key observation that our notion of c-clustered equation is equivalent to an abstract categorical form of equation earlier introduced by Milius and Urbat [18]. Our variety theorem for c-varieties of relational algebras then emerges by combining this equivalence with their Abstract Variety Theorem (recalled in Section 4.1). The categorical perspective has several advantages; most importantly, it underlines that c-clustered equations, and the corresponding c-reflexive quotients featuring in the closure properties of varieties, are not an ad-hoc concept but naturally arise from general principles. Moreover, it allows us to isolate the generic parts of the proof of the variety theorem from arguments specific to the particular setting.

While the main focus of our paper is on the model theory of equations with relational features, we also provide first steps towards a complete deduction system for such equations. In this regard, observe that the completeness of Birkhoff's classical equational logic [7] can be derived as an easy consequence of the exactness property of Σ-algebras, namely the fact that quotients of an algebra A can be represented as congruence relations on A, which are equivalence relations respected by all the operations. We establish a corresponding exactness result for algebras over relational structures, which yields a full characterization of quotient algebras in terms of suitable relations (Theorem 5.3). This turns out to be substantially more involved than the classical case, making it a result of independent interest, and we expect that it can serve as a basis for a complete equational logic in our present setting; see also the discussion in Section 6.

Proof details omitted due to space restrictions can be found in the arXiv v2 version [12] of our paper.

2 Preliminaries

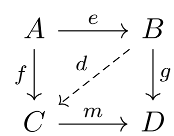
We assume that readers are familiar with notions of basic category theory such as functors, (co)limits, and adjunctions. For a gentle introduction, see Mac Lane's book [14].

Let us briefly recall some categorical terminology we use in the sequel. A *factorization system* $(\mathcal{E}, \mathcal{M})$ in a category $\mathscr{A}$ consists of two classes $\mathcal{E}$ and $\mathcal{M}$ of morphisms in $\mathscr{A}$ such that

(1) both $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms and are closed under composition;

(2) every morphism f has a factorization $f = m \cdot e$ where $e \in \mathcal{E}$ and $m \in \mathcal{M}$;

(3) the *diagonal fill-in* property holds: for every commutative square as shown below where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique morphism d making both triangles commute.



The factorization system is *proper* if all morphisms in $\mathcal{E}$ are epic and all morphisms in $\mathcal{M}$ are monic; in this case, morphisms in $\mathcal{E}$ and $\mathcal{M}$ are denoted by $\twoheadrightarrow$ and $\rightarrowtail$, respectively. A simple example is the proper factorization system of Set, the category of sets and functions, given by $(\mathcal{E}, \mathcal{M}) =$ (surjective maps, injective maps). Given a proper factorization system $(\mathcal{E}, \mathcal{M})$, a *quotient* of an object A is represented by a morphism $e\colon A \twoheadrightarrow B$ in $\mathcal{E}$ and a *subobject* by a morphism $m\colon B \rightarrowtail A$ in $\mathcal{M}$. Two quotients $e\colon A \twoheadrightarrow B$ and $e'\colon A \twoheadrightarrow B'$ are identified if there exists an isomorphism $i\colon B \xrightarrow{\cong} B'$ such that $e' = i \cdot e$; dually for subobjects. The category $\mathscr{A}$ is *$\mathcal{E}$-co-well-powered* if for every object A the class of quotients of A, taken up to isomorphism, forms a small set.

An object $X \in \mathscr{A}$ is called *projective* w.r.t. a morphism $e\colon A \to B$ if the induced map $\mathscr{A}(X, e) = e \cdot (-)\colon \mathscr{A}(X, A) \to \mathscr{A}(X, B)$ between hom-sets is surjective. In other words, for every $h\colon X \to B$, there exists a morphism $g\colon X \to A$ such that $h = e \cdot g$.

3 Algebras over Relational Structures

In the following we study algebraic structures whose underlying set is equipped with additional relations, which the operations of the algebra respect in a user-defined manner.

A *(finitary) relational signature* $\mathscr{S}$ is a set of *relation symbols* with associated positive *arity* $\mathsf{ar}(R) \in \mathbb{N}_+$ for each $R \in \mathscr{S}$. An *$\mathscr{S}$-structure* $(A, (R_A)_{R \in \mathscr{S}})$ is given by a set A equipped with an n-ary relation $R_A \subseteq A^n$ for every n-ary relation symbol $R \in \mathscr{S}$. We usually just write A for $(A, (R_A)_{R \in \mathscr{S}})$. A *morphism* $h\colon A \to B$ of $\mathscr{S}$-structures is a relation-preserving map: for each n-ary $R \in \mathscr{S}$ and $a_1, \ldots, a_n \in A$,

$$R_A(a_1, \ldots, a_n) \quad \Longrightarrow \quad R_B(h(a_1), \ldots, h(a_n)).$$

Conversely, a map $h\colon A \to B$ is said to *reflect relations* if for each n-ary $R \in \mathscr{S}$ and $a_1, \ldots, a_n \in A$,

$$R_A(a_1, \ldots, a_n) \quad \Longleftarrow \quad R_B(h(a_1), \ldots, h(a_n)).$$

An *embedding* is an injective map $m\colon A \rightarrowtail B$ that both preserves and reflects relations.

We denote the category of $\mathscr{S}$-structures and their morphisms by $\mathsf{Str}(\mathscr{S})$, and its forgetful functor by

$$U\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Set}.$$

For every $A \in \mathsf{Str}(\mathscr{S})$ we write $|A|$ for the cardinality of its underlying set UA.

Remark 3.1 (1) The category $\mathsf{Str}(\mathscr{S})$ is complete and cocomplete, with limits and colimits formed at the level of underlying sets; in particular, U preserves limits and colimits. Specifically:

(a) The product $A = \prod_{i\in I} A_i$ is given by the cartesian product equipped with the relations defined by

$$R_A((a_{i,1})_{i\in I}, \dots, (a_{i,n})_{i\in I}) \quad\iff\quad \forall i \in I : R_{A_i}(a_{i,1}, \dots, a_{i,n}).$$

(b) The coproduct $A = \coprod_{i\in I} A_i$ is given by the disjoint union, and $R_A(a_1, \dots, a_n)$ holds iff $a_1, \dots, a_n$ lie in the same coproduct component A_i and $R_{A_i}(a_1, \dots, a_n)$.

(c) A diagram $D\colon I \to \mathsf{Str}(\mathscr{S})$ is *κ-directed*, for a regular cardinal number κ, if its scheme I is a κ-directed poset, that is, every subset of I of cardinality less than κ has an upper bound. A *κ-directed colimit* is a colimit of a κ-directed diagram. A *κ-directed union of embeddings* is a κ-directed colimit where all connecting morphisms $D_i \to D_j$ $(i \le j)$ are embeddings. To form the colimit of any κ-directed diagram D, one takes the colimit cocone $c_i\colon UD_i \to C$ $(i \in I)$ of UD in Set and equips C with the following relations for each n-ary relation symbol $R \in \mathscr{S}$:

$$R_C(x_1, \dots, x_n) \quad\iff\quad \exists i \in I.\, \exists y_1, \dots, y_n \in D_i.\; x_i = c_i(y_i) \,\wedge\, R_{D_i}(y_1, \dots, y_n)$$

In the case of a κ-directed union, C is the union of the sets UD_i $(i \in I)$ and all colimit injections c_i are embeddings. Moreover, if $z_i\colon D_i \to Z$ $(i \in I)$ is another cocone over D where all z_i are embeddings, then the unique mediating map $z\colon C \to Z$ such that $z_i = z \cdot c_i$ for all $i \in I$ is an embedding, too.

(2) The category $\mathsf{Str}(\mathscr{S})$ has the factorization system given by surjective morphisms and embeddings. Accordingly, *quotients* and *substructures* of $\mathscr{S}$-structures are represented by surjections and embeddings.

In the following we shall consider structures axiomatized by (possibly infinitary) Horn clauses:

Definition 3.2 An *infinitary Horn clause* over a set X of variables is an expression of either of the types

$$R_i(x_{i,1}, \dots, x_{i,n_i}) \;\; (i \in I) \;\;\vdash\;\; R(x_1, \dots, x_n), \tag{3.1}$$
$$R_i(x_{i,1}, \dots, x_{i,n_i}) \;\; (i \in I) \;\;\vdash\;\; x_1 = x_2, \tag{3.2}$$

where (a) I is a set, (b) $x_k, x_{i,k} \in X$ for all indices i, k, and (c) R_i $(i \in I)$ and R are relation symbols in $\mathscr{S}$ with arities n_i and n, respectively.

Definition 3.3 Let A be an $\mathscr{S}$-structure.

(1) The structure A *satisfies* the clause (3.1) if for every map[7] $h\colon X \to A$,

$$(R_i)_A(h(x_{i,1}), \dots, h(x_{i,n_i})) \text{ for all } i \in I \qquad\text{implies}\qquad R_A(h(x_1), \dots, h(x_n)).$$

(2) Similarly, A *satisfies* the clause (3.2) if for every map $h\colon X \to A$,

$$(R_i)_A(h(x_{i,1}), \dots, h(x_{i,n_i})) \text{ for all } i \in I \qquad\text{implies}\qquad h(x_1) = h(x_2).$$

Notation 3.4 From now on, we fix a relational signature $\mathscr{S}$ and a set Ax of infinitary Horn clauses over $\mathscr{S}$. We denote the full subcategory of structures satisfying all clauses in Ax by

$$\mathscr{C} \hookrightarrow \mathsf{Str}(\mathscr{S}).$$

[7] The map h can be thought of as an assignment of values in A to each variable in X.

Lemma 3.5 *The category $\mathscr{C}$ is closed under products and substructures in $\mathsf{Str}(\mathscr{S})$.*

Example 3.6 Our leading example is that of generalized metric spaces [20]. A *fuzzy relation* on a set A is a map $d\colon A \times A \to [0,1]$. Let $\mathsf{Ax}_{\mathrm{GM}}$ be a fixed subset of the following axioms:

$$\forall a \in A : d(a,a) = 0 \tag{Refl}$$
$$\forall a, b \in A : d(a,b) = 0 \implies a = b \tag{Pos}$$
$$\forall a, b \in A : d(a,b) = d(b,a) \tag{Sym}$$
$$\forall a, b, c \in A : d(a,c) \le d(a,b) + d(b,c) \tag{Tri}$$
$$\forall a, b, c \in A : d(a,c) \le \max\{d(a,b), d(b,c)\} \tag{Max}$$

A *generalized metric space* is a set A with a fuzzy relation $d_A\colon A \times A \to [0,1]$, subject to the axioms in $\mathsf{Ax}_{\mathrm{GM}}$. A map $h\colon A \to B$ between generalized metric spaces is *nonexpansive* if $d_B(h(a), h(a')) \le d_A(a,a')$ for $a, a' \in A$. We let GMet denote the category of generalized metric spaces and nonexpansive maps.[8]

We can regard generalized metric spaces as relational structures as follows. Consider the relational signature $\mathscr{S} = \{\, =_\varepsilon \colon \varepsilon \in [0,1] \,\}$ where $\mathsf{ar}(=_\varepsilon) = 2$ for each $\varepsilon \in [0,1]$. Let Ax be the corresponding[9] subset of the following Horn clauses, where $\varepsilon, \varepsilon' \in [0,1]$:

$$\vdash\ x =_0 x \tag{Refl$'$}$$
$$x =_0 y\ \vdash\ x = y \tag{Pos$'$}$$
$$x =_\varepsilon y\ \vdash\ y =_\varepsilon x \tag{Sym$'$}$$
$$x =_\varepsilon y,\, y =_{\varepsilon'} z\ \vdash\ x =_{\varepsilon+\varepsilon'} z \qquad (\varepsilon + \varepsilon' \le 1) \tag{Tri$'$}$$
$$x =_\varepsilon y,\, y =_{\varepsilon'} z\ \vdash\ x =_{\max\{\varepsilon,\varepsilon'\}} z \tag{Max$'$}$$
$$x =_\varepsilon y\ \vdash\ x =_{\varepsilon'} y \qquad (\varepsilon < \varepsilon') \tag{Up}$$
$$x =_{\varepsilon'} y\ \ (\varepsilon' > \varepsilon)\ \vdash\ x =_\varepsilon y \tag{Arch}$$

An $\mathscr{S}$-structure $(A, (=_\varepsilon)_{\varepsilon\in[0,1]})$ satisfying Ax then gives rise to a generalized metric space (A,d) with the generalized metric defined by $d(a,a') := \inf\{\varepsilon : a =_\varepsilon a'\}$. In the opposite direction, a generalized metric space (A,d) defines an $\mathscr{S}$-structure $(A, (=_\varepsilon)_{\varepsilon\in[0,1]})$ where $a =_\varepsilon a'$ holds iff $d(a,a') \le \varepsilon$. This $\mathscr{S}$-structure clearly satisfies Ax. Moreover, these two correspondences are mutually inverse:

(1) Consider the composite $(A,d) \mapsto (A, (=_\varepsilon)_{\varepsilon\in[0,1]}) \mapsto (A, d')$. Then we clearly have $d(a,a') = \inf\{\varepsilon : a =_\varepsilon a'\}$ by the definition of $=_\varepsilon$. Thus $d(a,a') = d'(a,a')$.

(2) Consider the composite $(A, (=_\varepsilon)_{\varepsilon\in[0,1]}) \mapsto (A,d) \mapsto (A, (='_\varepsilon)_{\varepsilon\in[0,1]})$. Then we have that $a =_\varepsilon a'$ implies $d(a,a') \le \varepsilon$, which implies $a ='_\varepsilon a'$. Conversely, if $a ='_\varepsilon a'$, then $d(a,a') \le \varepsilon$, and thus for each $\varepsilon' > \varepsilon$ we have $d(a,a') < \varepsilon'$. Since $d(a,a') = \inf\{\delta : a =_\delta a'\}$, there exists $\delta < \varepsilon'$ such that $a =_\delta a'$. By using (Up) we see that $a =_{\varepsilon'} a'$. This holds for each $\varepsilon' > \varepsilon$, hence (Arch) yields $a =_\varepsilon a'$.

Furthermore, nonexpansive maps and morphisms of $\mathscr{C}$ are clearly in one-to-one correspondence. Consequently, the category GMet is isomorphic to the category $\mathscr{C}$. For the case of (ordinary) metric spaces, where $\mathsf{Ax}_{\mathrm{GM}}$ consists of (Refl), (Pos), (Sym), (Tri), this was already observed by Mardare et al. [16].

Example 3.7 We mention some further examples of categories of relational structures specified by infinitary Horn clauses.

(1) The category Set of sets and functions is specified by the empty relational signature and the empty set of axioms.

(2) The category Pos of partially ordered sets (posets) and monotone maps is specified by the relational

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[8] Since GMet is parametric in the choice of $\mathsf{Ax}_{\mathrm{GM}}$, this defines a family of categories.

[9] This means that Ax contains (Up) and (Arch), and a primed axiom appears in Ax iff the corresponding non-primed axiom appears in $\mathsf{Ax}_{\mathrm{GM}}$.

signature $\mathscr{S}$ consisting of a single binary relation symbol $\leq$ and the axioms

$$\vdash\ x \leq x, \qquad x \leq y,\, y \leq z\ \vdash\ x \leq z, \qquad \text{and} \qquad x \leq y,\, y \leq x\ \vdash\ x = y.$$

(3) Let L be a complete lattice where for every $l \in L$ and $P \subseteq L$ with $l > \bigwedge P$ one has $l \geq p$ for some $p \in P$. Moreover, let $\mathscr{S}$ be the relational signature consisting of binary relation symbols $=_l$ for all $l \in L$ and consider the axioms

$$x =_l y\ \vdash\ x =_{l'} y \qquad (l < l') \tag{Up}$$

$$x =_{l'} y\ \ (l' > l)\ \vdash\ x =_l y \tag{Arch}$$

This specifies the category of *L-valued relations*: its objects are sets X equipped with a map $P\colon X \times X \to L$, and its morphisms $(X, P) \to (Y, Q)$ are maps $h\colon X \to Y$ such that $Q(h(x), h(y)) \leq P(x, y)$. Of course, fuzzy relations are the special case $L = [0, 1]$.

(4) A signature of partial operations is a set P of operation symbols f with prescribed arities $\mathsf{ar}(f) \in \mathbb{N}$. A (partial) P-algebra is given by a set A equipped with a partial map $f_A\colon A^{\mathsf{ar}(f)} \to A$ for each $f \in P$. A morphism of partial algebras is a (total) map $h\colon A \to B$ such that whenever $f_A(x_1, \ldots, x_{\mathsf{ar}(f)})$ is defined, then $f_B(h(x_1), \ldots, h(x_{\mathsf{ar}(f)}))$ is defined and equals $h(f_A(x_1, \ldots, x_{\mathsf{ar}(f)}))$. The category of partial P-algebras and their morphims is isomorphic to the category specified by the relational signature consisting of relational symbols α_f of arity $\mathsf{ar}(f) + 1$ for all $f \in P$ (with $\alpha_f(x_1, \ldots, x_{\mathsf{ar}(f)}, y)$ being understood as $f(x_1, \ldots, x_{\mathsf{ar}(f)}) = y$), and the axioms $\alpha_f(x_1, \ldots, x_{\mathsf{ar}(f)}, y), \alpha_f(x_1, \ldots, x_{\mathsf{ar}(f)}, z)\ \vdash\ y = z$.

Next, we introduce *lifted algebraic signatures* over relational structures, which extends the corresponding notion by Mio et al. [20] for the setting of generalized metric spaces.

Definition 3.8 A functor $G\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Str}(\mathscr{S})$ is a *lifting* of $F\colon \mathsf{Set} \to \mathsf{Set}$ if the square below commutes:

$$\begin{array}{ccc} \mathsf{Str}(\mathscr{S}) & \xrightarrow{\ G\ } & \mathsf{Str}(\mathscr{S}) \\ {\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle U} \\ \mathsf{Set} & \xrightarrow{\ F\ } & \mathsf{Set} \end{array}$$

Definition 3.9 An *(infinitary) algebraic signature* is a set Σ of operation symbols σ with prescribed arity $\mathsf{ar}(\sigma)$, a cardinal number. A *lifted algebraic signature* $\widehat{\Sigma}$ is given by a signature Σ together with a lifting $L_\sigma\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Str}(\mathscr{S})$ of the functor $(-)^n\colon \mathsf{Set} \to \mathsf{Set}$ for every n-ary operation symbol $\sigma \in \Sigma$. Given $A \in \mathsf{Str}(\mathscr{S})$ we write $L_\sigma(R_A)$ for the interpretation of the relation symbol $R \in \mathscr{S}$ in the structure $L_\sigma(A)$:

$$L_\sigma(A) = (A^n, (L_\sigma(R_A))_{R \in \mathscr{S}}).$$

Assumption 3.10 In the remainder of the paper we fix a lifted algebraic signature $\widehat{\Sigma}$ with associated lifted functors L_σ $(\sigma \in \Sigma)$. We assume that each L_σ preserves embeddings. Moreover, we choose a regular cardinal κ such that every operation symbol in Σ has arity $< \kappa$; hence Σ is a *κ-ary signature.*

Definition 3.11 A *$\widehat{\Sigma}$-algebra* is given by an $\mathscr{S}$-structure A equipped with n-ary operations

$$\sigma_A\colon (A^n, (L_\sigma(R_A))_{R \in \mathscr{S}}) \to (A, (R_A)_{R \in \mathscr{S}}) \qquad \text{in} \quad \mathsf{Str}(\mathscr{S})$$

for every n-ary operation symbol $\sigma \in \Sigma$. A *morphism* $h\colon A \to B$ of $\widehat{\Sigma}$-algebras is a map from A to B that is both a $\mathsf{Str}(\mathscr{S})$-morphism and a Σ-algebra morphism; the latter means that $h(\sigma_A(a_1, \ldots, a_n)) = \sigma_B(h(a_1), \ldots, h(a_n))$ for each n-ary operation symbol $\sigma \in \Sigma$ and $a_1, \ldots, a_n$. We let $\mathbf{Alg}(\widehat{\Sigma})$ denote the category of $\widehat{\Sigma}$-algebras and their morphisms, and $\mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$ the full subcategory of *$\widehat{\Sigma}$-algebras over $\mathscr{C}$*, that is, $\widehat{\Sigma}$-algebras whose underlying $\mathscr{S}$-structure lies in the full subcategory $\mathscr{C} \hookrightarrow \mathsf{Str}(\mathscr{S})$ given by Ax.

The use of lifted signatures allows for some flexibility in how the individual operations of an algebra respect the underlying relations. This is illustrated by the following examples.

Example 3.12 (1) For every relational signature $\mathscr{S}$, there are two simple choices of a lifting $L_\sigma\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Str}(\mathscr{S})$ for an n-ary operation symbol $\sigma \in \Sigma$:

(a) The *discrete lifting* L_σ^{disc} maps $A \in \mathsf{Str}(\mathscr{S})$ to A^n equipped with empty relations. Then the operation $\sigma_A\colon A^n \to A$ of a $\widehat{\Sigma}$-algebra A is just an arbitrary homomorphism of Σ-algebras that is not subject to any relational conditions.

(b) The *product lifting* L_σ^{prod} maps $A \in \mathsf{Str}(\mathscr{S})$ to the product structure A^n in $\mathsf{Str}(\mathscr{S})$. Then the operation $\sigma_A\colon A^n \to A$ is relation-preserving w.r.t. the product structure (Remark 3.1).

(2) For the signature $\mathscr{S} = \{=_\varepsilon\colon \varepsilon \in [0,1]\}$ and $\mathscr{C} = \mathsf{GMet}$ we obtain the quantitative $\widehat{\Sigma}$-algebras by Mio et al. [20]. In *op. cit.* and in [21] the authors consider two non-trivial liftings which are motivated by applications in quantitative term rewriting [11] and machine learning [9]:

(a) The *Lipschitz lifting* $L_\sigma^{\mathsf{Lip},\alpha}$ for a fixed parameter $\alpha \in [1,\infty)$ maps $A \in \mathsf{Str}(\mathscr{S})$ to A^n equipped with the relations $(a_i)_{i<n} =_\varepsilon (a_i')_{i<n}$ iff $a_i =_{\varepsilon/\alpha} a_i'$ for all $i < n$. Then the operation $\sigma_A\colon A^n \to A$ of a quantitative $\widehat{\Sigma}$-algebra A is an α-Lipschitz map w.r.t. the product metric d on A^n, which is defined by $d((a_i)_{i<n},(a_i')_{i<n}) := \sup_{i<n} d_A(a_i,a_i')$.

(b) The *Łukaszyk–Karmowski* lifting $L_\sigma^{\mathsf{LK},p}$, for a fixed parameter $p \in (0,1)$ and a binary operation symbol $\sigma \in \Sigma$, sends $A \in \mathsf{Str}(\mathscr{S})$ to A^2 equipped with the relations defined by $(a_1,a_2) =_\varepsilon (a_1',a_2')$ iff there exist $\varepsilon_{ij} \in [0,1]$ $(i,j = 1,2)$ such that $a_1 =_{\varepsilon_{11}} a_1'$, $a_1 =_{\varepsilon_{12}} a_2'$, $a_2 =_{\varepsilon_{21}} a_1'$, $a_2 =_{\varepsilon_{22}} a_2'$ and $\varepsilon = p^2\varepsilon_{11} + p(1-p)\varepsilon_{12} + (1-p)p\varepsilon_{21} + (1-p)^2\varepsilon_{22}$. Then given a quantitative $\widehat{\Sigma}$-algebra A the operation $\sigma_A\colon A^2 \to A$ is nonexpansive w.r.t. the Łukaszyk–Karmowski distance [13].

We note that the above liftings restrict to $L_\sigma\colon \mathsf{GMet} \to \mathsf{GMet}$ for suitable choices of $\mathsf{Ax}_{\mathrm{GM}}$. This is the type of lifting studied by Mio et al. [20].

(3) For the signature $\mathscr{S} = \{\leq\}$ and $\mathscr{C} = \mathsf{Pos}$ we obtain various notions of *ordered algebras*, i.e. algebras carried by a poset.

(a) The discrete lifting and the product lifting correspond to ordered algebras with arbitrary or monotone operations, respectively. The latter are standard ordered algebras studied in the literature [8].

(b) These two liftings admit a common generalization: for a fixed subset $S \subseteq \{1,\dots,n\}$ and $\sigma \in \Sigma$, let L_σ^S be the lifting that sends $A \in \mathsf{Str}(\mathscr{S})$ to A^n with the relation $(a_i)_{i<n} \leq (a_i')_{i<n}$ iff $a_i \leq a_i'$ for every $i \in S$. An operation $\sigma_A\colon A^n \to A$ is then monotone in precisely the coordinates from S.

(c) The *lexicographic lifting* L_σ^{lex} sends $A \in \mathsf{Str}(\mathscr{S})$ to A^n with $(a_i)_{i<n} \leq (a_i')_{i<n}$ if either $(a_i)_{i<n} = (a_i')_{i<n}$, or $a_k \leq a_k'$ for $k = \min\{i < n : a_i \neq a_i'\}$. An operation $\sigma_A\colon A^n \to A$ is then monotone w.r.t. the lexicographic ordering on A^n.

Furthermore, combinations of the above items are easily conceivable, e.g. we may specify ordered algebras with a monotone operation $\sigma_A\colon A^5 \to A$ where the order on A^5 is lexicographic in the first two coordinates, coordinatewise in the last two, and discrete in the third coordinate.

Remark 3.13 Since coproducts in $\mathsf{Str}(\mathscr{S})$ are formed at the level of underlying sets, the polynomial endofunctor $H_\Sigma = \coprod_{\sigma\in\Sigma}(-)^{\mathsf{ar}(\sigma)}$ on Set associated to the algebraic signature Σ lifts to the endofunctor $H_{\widehat{\Sigma}} = \coprod_{\sigma\in\Sigma} L_\sigma$ on $\mathsf{Str}(\mathscr{S})$, and the category $\mathbf{Alg}(\widehat{\Sigma})$ is isomorphic to the category of algebras for $H_{\widehat{\Sigma}}$.

The next three lemmas establish some simple properties of the category $\mathbf{Alg}(\widehat{\Sigma})$.

Lemma 3.14 *The categories* $\mathbf{Alg}(\widehat{\Sigma})$ *and* $\mathbf{Alg}(\mathscr{C},\widehat{\Sigma})$ *have products.*

Proof sketch. This is immediate from Remark 3.13 and the well-known fact that for every endofunctor $H\colon \mathscr{A} \to \mathscr{A}$, the forgetful functor from the category of H-algebras to $\mathscr{A}$ creates limits. More explicitly, the product of algebras A_j in $\mathbf{Alg}(\widehat{\Sigma})$, $j \in J$, is given by their product $\mathscr{S}$-structure $A = \prod_{j\in J} A_j$ (Remark 3.1) with operations defined coordinatewise. The product in $\mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$ is formed in the same way, using that $\mathscr{C}$ is closed under products in $\mathsf{Str}(\mathscr{S})$ by Lemma 3.5. ◻

Lemma 3.15 (Homomorphism Theorem) *Let $e\colon A \twoheadrightarrow B$ and $h\colon A \to C$ be $\widehat{\Sigma}$-algebra morphisms with e surjective. Then h factorizes through e iff the following conditions hold:*

(1) *for every $a, a' \in A$, if $e(a) = e(a')$, then $h(a) = h(a')$;*

(2) *for every n-ary $R \in \mathscr{S}$ and $a_1, \dots, a_n \in A$, if $R_B(e(a_1), \dots e(a_n))$, then $R_C(h(a_1), \dots, h(a_n))$.*

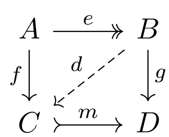
Proof. The 'only if' statement is clear. For the 'if' statement the first condition asserts that there exists a unique map $g\colon B \to C$ such that $h = g \cdot e$. Since e is surjective, g forms a Σ-algebra morphism, and by the second condition g is also a $\mathsf{Str}(\mathscr{S})$-morphism. Hence, g is a $\widehat{\Sigma}$-algebra morphism. ◻

Lemma 3.16 *The category $\mathbf{Alg}(\widehat{\Sigma})$ has a proper factorization system given by $\widehat{\Sigma}$-algebra morphisms carried by surjections and embeddings.*

Proof. Every $\widehat{\Sigma}$-algebra morphism $h\colon A \to B$ admits a factorization into an surjection followed by an embedding:

$$h = \big(A \xrightarrow{e} h[A] \xrightarrow{m} B\big),$$

where $h[A] \subseteq B$ is the image of h (which forms a $\widehat{\Sigma}$-subalgebra of B), the morphism m is the inclusion map, and the morphism e is the codomain restriction of h. For the diagonal fill-in property, consider a square as shown below whose outside commutes, where e is surjective and m is an embedding:



The homomorphism theorem yields a unique $\widehat{\Sigma}$-algebra morphism d making the upper triangle commutative. Since e is surjective, this implies that the lower triangle also commutes. ◻

We conclude this section with the construction of free $\widehat{\Sigma}$-algebras.

Lemma 3.17 *The functor $H_{\widehat{\Sigma}}$ preserves κ-directed unions of embeddings.*

Proof. Since coproducts in $\mathsf{Str}(\mathscr{S})$ preserve embeddings and commute with κ-directed colimits, it suffices to show that each L_σ preserves κ-directed unions of embeddings. (Recall that L_σ preserves embeddings by Assumption 3.10.) We know that for every $n < \kappa$, the set functor $(-)^n$ preserves κ-directed unions because κ-directed colimits in Set commute with κ-small limits (in particular products with less than κ factors). Moreover, from Remark 3.1(c) we know that given a cocone $c_i\colon D_i \rightarrowtail C$ ($i \in I$) of embeddings over a κ-directed diagram of embeddings, then $(c_i)_{i\in I}$ is a colimit cocone of D iff $(Uc_i)_{i\in I}$ is a colimit cocone of UD, where $U\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Set}$ is the forgetful functor. The desired result now follows since for every $\sigma \in \Sigma$ of arity n we have $U \cdot L_\sigma = (-)^n \cdot U$ (cf. Definition 3.8) and since $(-)^n \cdot U$ preserves κ-directed unions of embeddings.

◻

Proposition 3.18 *The forgetful functor from $\mathbf{Alg}(\widehat{\Sigma})$ to $\mathsf{Str}(\mathscr{S})$ has a left adjoint assigning to every $\mathscr{S}$-structure X the* free $\widehat{\Sigma}$-algebra $T_{\widehat{\Sigma}}X$ *on X. Its underlying Σ-algebra is the free Σ-algebra $T_\Sigma X$ on (the underlying set of) X, carried by the set of all well-founded Σ-trees over the set X.*

In more detail, a Σ-*tree over* X is a possibly infinite ordered tree with nodes labelled in $\Sigma \cup X$, where each node labelled by $\sigma \in \Sigma$ has exactly $\mathsf{ar}(\sigma)$ successors and each X-labelled node is a leaf. A Σ-tree is *well-founded* if it contains no infinite path. If the signature Σ is finitary (i.e. $\kappa = \omega$), a well-founded Σ-tree is precisely the (finite) syntax tree of a Σ-term in the usual sense.

Remark 3.19 The proof makes use of a well-known construction, due to Adámek [1], of free algebras for an endofunctor $H\colon \mathscr{A} \to \mathscr{A}$ on a cocomplete category $\mathscr{A}$. Given an object X of $\mathscr{A}$, the *free-algebra chain of* X for H is the ordinal-indexed chain of objects $X_i^\sharp$ and connecting morphisms $w_{i,j}\colon X_i^\sharp \to X_j^\sharp$ $(i < j)$ defined as follows by transfinite recursion:

$$
\begin{aligned}
X_0^\sharp &= X, \\
X_{j+1}^\sharp &= HX_j^\sharp + X && \text{for all ordinals } j, \\
X_j^\sharp &= \mathsf{colim}_{i<j} X_i^\sharp && \text{for all limit ordinals } j, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
w_{0,1} &= \big(X_0^\sharp = X \xrightarrow{\mathsf{inr}} HX_0^\sharp + X = X_1^\sharp\big) && \text{is the right-hand coproduct injection,} \\
w_{j+1,k+1} &= \big(X_{j+1}^\sharp = HX_j^\sharp + X \xrightarrow{Hw_{j,k}+\mathsf{id}_X} HX_k^\sharp + X = X_{k+1}^\sharp\big),
\end{aligned}
$$

$w_{i,j}$ $(i < j)$ is the colimit cocone for limit ordinals j,

$w_{j,j+1}$ (j limit ordinal) is the unique morphism such that $w_{j,j+1} \cdot w_{i+1,j} = w_{i+1,j+1}$ for $i < j$.

The free-algebra chain is said to *converge in* λ *steps* if $w_{\lambda,\lambda+1}$ is an isomorphism. In this case, the inverse $w_{\lambda,\lambda+1}^{-1}\colon HX_\lambda^\sharp + X \to X_\lambda^\sharp$ yields, by precomposing with the two coproduct injections, the structure and universal morphism of a free H-algebra on X. Note that if the functor H preserves the colimit formed at some limit step λ, then the free-algebra chain converges in λ steps.

Proof of Proposition 3.18. Since Σ is a κ-ary signature, the polynomial set functor H_Σ preserves κ-directed colimits. Hence, for every set X, the free-algebra chain of X for H_Σ converges in κ steps. Moreover, it is well-known that the free Σ-algebra on X is carried by the set of all well-founded Σ-trees over X (see e.g. [5, Thm. 2.9]). Using that H_Σ preserves injections and that they are closed under coproducts, an easy transfinite induction shows that each connecting map $w_{i,j}$ is injective and that the colimits at limit ordinals are unions.

Similarly, for every $\mathscr{S}$-structure X, the free-algebra chain of X for $H_{\widehat{\Sigma}}$ is formed by embeddings (using Remark 3.1 and that each L_σ preserves embeddings, see Assumption 3.10), taking unions at limit ordinals. From Lemma 3.17 we know that the functor $H_{\widehat{\Sigma}}$ preserves κ-directed unions of embeddings. Hence, the free-algebra chain of X for $H_{\widehat{\Sigma}}$ converges in κ steps to the free $\widehat{\Sigma}$-algebra on X, in symbols: $X_\kappa^\sharp = T_{\widehat{\Sigma}}X$. Moreover, since the forgetful functor $U\colon \mathsf{Str}(\mathscr{S}) \to \mathsf{Set}$ preserves κ-directed colimits, we see that it maps the free-algebra chain of X for $H_{\widehat{\Sigma}}$ to the free-algebra chain of UX for H_Σ. In particular, $U(T_{\widehat{\Sigma}}X)$ is the set of all well-founded Σ-trees over X. ◻

4 Variety Theorems

In this section we establish the variety theorem for $\widehat{\Sigma}$-algebras over $\mathscr{C}$, our fixed subcategory of $\mathsf{Str}(\mathscr{S})$. Rather than stating and proving the theorem from scratch, we will take a more principled approach and present it as an instance of a general categorical perspective on equations and varieties.

4.1 Abstract Varieties

We first review the abstract variety theorem by Milius and Urbat [18], which characterizes classes of objects in a category axiomatizable by an abstract notion of equation. We state the theorem in a slightly simplified form that is sufficient for our intended application to algebras over relational structures.

Fix a category $\mathscr{A}$ with a proper factorization system $(\mathcal{E}, \mathcal{M})$, a full subcategory $\mathscr{A}_0 \hookrightarrow \mathscr{A}$, and a class $\mathscr{X}$ of objects of $\mathscr{A}$. Informally, we think of $\mathscr{A}$ as a category of algebraic structures, of $\mathscr{A}_0$ as the subcategory of those algebras over which varieties are formed, and of $\mathscr{X}$ as the class of term algebras over which equations are formed; see Example 4.5 below for a simple instantiation. The class $\mathscr{X}$ determines a class of quotients in $\mathscr{A}$ defined by

$$\mathcal{E}_{\mathscr{X}} = \{\, e \in \mathcal{E} : \text{every } X \in \mathscr{X} \text{ is projective w.r.t. } e \,\}. \tag{4.1}$$

An $\mathcal{E}_{\mathscr{X}}$*-quotient* is a quotient represented by a morphism in $\mathcal{E}_{\mathscr{X}}$.

Remark 4.1 In order to determine $\mathcal{E}_{\mathscr{X}}$ in a category $\mathscr{A}$ of algebras with additional structure, it suffices to look at the category of underlying structures. Specifically, suppose that

(1) the category $\mathscr{A}$ is part of an adjoint situation $F \dashv U \colon \mathscr{A} \to \mathscr{B}$,

(2) there is a class $\mathscr{X}'$ objects in $\mathscr{B}$ such that $\mathscr{X} = \{\, FX' \,:\, X' \in \mathscr{X}' \,\}$, and

(3) there is a class $\mathcal{E}'$ of morphisms in $\mathscr{B}$ such that $\mathcal{E} = \{\, e \in \mathscr{A} \,:\, Ue \in \mathcal{E}' \,\}$.

In analogy to $\mathcal{E}_{\mathscr{X}}$, we define the following class $\mathcal{E}'_{\mathscr{X}'}$ of morphisms in $\mathscr{B}$:

$$\mathcal{E}'_{\mathscr{X}'} = \{\, e' \in \mathcal{E}' : \text{every } X' \in \mathscr{X}' \text{ is projective w.r.t. } e' \,\}.$$

Then the class $\mathcal{E}_{\mathscr{X}}$ is given by $\mathcal{E}_{\mathscr{X}} = \{\, e \in \mathcal{E} \,:\, Ue \in \mathcal{E}'_{\mathscr{X}'} \,\}$. Indeed, for all $e \in \mathcal{E}$, one has

$$\begin{aligned} e \in \mathcal{E}_{\mathscr{X}} &\iff \forall X \in \mathscr{X} : \mathscr{A}(X, e) \text{ is surjective} \\ &\iff \forall X' \in \mathscr{X}' : \mathscr{A}(FX', e) \text{ is surjective} \\ &\iff \forall X' \in \mathscr{X}' : \mathscr{B}(X', Ue) \text{ is surjective} \\ &\iff Ue \in \mathcal{E}'_{\mathscr{X}'}. \end{aligned}$$

Definition 4.2 (1) An *abstract equation* is an $\mathcal{E}$-morphism $e \colon X \twoheadrightarrow E$ where $X \in \mathscr{X}$ and $E \in \mathscr{A}_0$.

(2) An object $A \in \mathscr{A}_0$ *satisfies* the abstract equation e if every morphism $h \colon X \to A$ factorizes through e, that is, $h = g \cdot e$ for some $g \colon E \to A$.

(3) Given a class $\mathbb{E}$ of abstract equations, we denote by $\mathcal{V}(\mathbb{E})$ the class of objects satisfying all equations in $\mathbb{E}$. A class $\mathcal{V}$ of objects of $\mathscr{A}_0$ is an *abstract variety* if it is axiomatizable by abstract equations, that is, $\mathcal{V} = \mathcal{V}(\mathbb{E})$ for some class $\mathbb{E}$ of equations.

The following theorem, which is a special case of a result by Milius and Urbat [18, Thm. 3.16], characterizes abstract varieties by their closure properties:

Theorem 4.3 (Abstract Variety Theorem) *Suppose that the category $\mathscr{A}$ is $\mathcal{E}$-co-well-powered and has products, that $\mathscr{A}_0 \hookrightarrow \mathscr{A}$ is closed under products and subobjects, and that every object of $\mathscr{A}_0$ is an $\mathcal{E}_{\mathscr{X}}$-quotient of some object of $\mathscr{X}$. Then for every class $\mathcal{V}$ of objects of $\mathscr{A}_0$,*

$\mathcal{V}$ is an abstract variety iff $\mathcal{V}$ is closed under $\mathcal{E}_{\mathscr{X}}$-quotients, subobjects, and products.

Remark 4.4 (1) Closure under $\mathcal{E}_{\mathscr{X}}$-quotients means that for every $\mathcal{E}_{\mathscr{X}}$-quotient $e \colon A \twoheadrightarrow B$ in $\mathscr{A}_0$, if $A \in \mathcal{V}$ then $B \in \mathcal{V}$. In particular, we assume $B \in \mathscr{A}_0$ from the outset.

(2) The key condition of Theorem 4.3 is the requirement that every object of $\mathscr{A}_0$ is an $\mathcal{E}_{\mathscr{X}}$-quotient of some object of $\mathscr{X}$. It captures, on an abstract categorical level, the intuition that the design of a concrete variety theorem needs to strike a balance: if one aims for expressive equations (corresponding to a 'large' choice of $\mathscr{X}$), one needs to make sure that the class $\mathcal{E}_{\mathscr{X}}$ remains rich enough.

Example 4.5 The classical Birkhoff Variety Theorem [7] corresponds to the instantiation

- $\mathscr{A} = \mathscr{A}_0 = \Sigma$-algebras for a finitary algebraic signature Σ;
- $(\mathcal{E}, \mathcal{M}) =$ (surjective, injective);
- $\mathscr{X} =$ all free (term) algebras $T_\Sigma X$, where X is a set of variables.

Note that $\mathcal{E}_{\mathscr{X}} = \mathcal{E}$ (hence varieties are closed under all quotients) and that an abstract equation $e\colon T_\Sigma X \twoheadrightarrow E$ can be presented via a set of ordinary equations between Σ-terms given by the kernel relation of e:

$$e \;\widehat{=}\; \{\, s = t : s, t \in T_\Sigma X,\, e(s) = e(t) \,\}.$$

Indeed, a Σ-algebra satisfies e iff it satisfies the above term equations in the usual sense.

4.2 Varieties of Algebras over Relational Structures

We now employ the above abstract framework to derive a variety theorem for algebras over relational structures. The variety theorem is parametric in a cardinal number $c > 1$, which determines the type of quotients under which varieties are closed. A structure $X \in \mathsf{Str}(\mathscr{S})$ is called *c-clustered* if it can be expressed as a coproduct $X = \coprod_{j\in J} X_j$ where $|X_j| < c$ for each $j \in J$. We instantiate the Abstract Variety Theorem 4.3 to the following data:

- $\mathscr{A} = \mathbf{Alg}(\widehat{\Sigma})$ and $\mathscr{A}_0 = \mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$ for a lifted signature $\widehat{\Sigma}$ satisfying Assumption 3.10;
- $(\mathcal{E}, \mathcal{M}) = (\text{surjections}, \text{embeddings})$, cf. Lemma 3.16;
- $\mathscr{X}$ = all free algebras $T_{\widehat{\Sigma}} X$ where $X \in \mathsf{Str}(\mathscr{S})$ is a c-clustered structure.

We first characterize the class $\mathcal{E}_{\mathscr{X}}$ as defined in (4.1). The characterization is based on a generalization of the notion of *c-reflexive morphism* by Mardare et al. [16] from metric spaces to relational structures:

Definition 4.6 A morphism $e\colon A \to B$ in $\mathsf{Str}(\mathscr{S})$ is *c-reflexive* if for every substructure $B_0 \subseteq B$ of cardinality $|B_0| < c$, there exists a substructure $A_0 \subseteq A$ such that e restricts to an isomorphism in $\mathsf{Str}(\mathscr{S})$ (i.e. a bijective embedding) $e_0\colon A_0 \xrightarrow{\cong} B_0$. If additionally e is surjective, then e is a *c-reflexive quotient*. By extension, a quotient in $\mathbf{Alg}(\widehat{\Sigma})$ is *c-reflexive* if its underlying quotient in $\mathsf{Str}(\mathscr{S})$ is c-reflexive.

Lemma 4.7 *The class $\mathcal{E}_{\mathscr{X}}$ consists of all c-reflexive quotients in $\mathbf{Alg}(\widehat{\Sigma})$.*

Proof. To prove Lemma 4.7, we apply Remark 4.1 to the adjunction $\mathbf{Alg}(\widehat{\Sigma}) \underset{\longleftarrow}{\overset{\top}{\longrightarrow}} \mathsf{Str}(\mathscr{S})$ and

$$\mathscr{X}' = c\text{-clustered structures} \qquad \text{and} \qquad \mathcal{E}' = \text{surjective } \mathsf{Str}(\mathscr{S})\text{-morphisms}.$$

It thus suffices to prove the characterization of $\mathcal{E}_{\mathscr{X}}$ for the case where the signature Σ is empty; that is, we can assume that $\mathscr{A} = \mathsf{Str}(\mathscr{S})$ and $\mathscr{X}$ = c-clustered structures.

Note that $\mathscr{X}$ is the closure of the class $\mathscr{X}_c = \{X \in \mathsf{Str}(\mathscr{S}) \;:\; |X| < c\}$ under coproducts. Since a coproduct is projective w.r.t. some morphism e if all of its coproduct components are, we have $\mathcal{E}_{\mathscr{X}} = \mathcal{E}_{\mathscr{X}_c}$. Hence, it suffices to show that, for every surjection $e\colon A \twoheadrightarrow B$ in $\mathsf{Str}(\mathscr{S})$,

$$e \in \mathcal{E}_{\mathscr{X}_c} \quad\iff\quad e \text{ is } c\text{-reflexive.}$$

($\Longrightarrow$) Suppose that $e \in \mathcal{E}_{\mathscr{X}_c}$, and let $m\colon B_0 \rightarrowtail B$ be a substructure of cardinality $< c$. Then $B_0 \in \mathscr{X}_c$, and thus, there exists $g\colon B_0 \to A$ such that $e \cdot g = m$. Let $A_0 = g[B_0]$. It follows that $e[A_0] = B_0$, whence $e\colon A_0 \to B_0$ is a surjection that preserves relations. It also reflects relations: for every n-ary relation symbol $R \in \mathscr{S}$ and $g(b_1), \ldots, g(b_n) \in A_0$, we have

$$\begin{aligned} R_{B_0}(e(g(b_1)), \ldots, e(g(b_n))) &\iff R_B(m(b_1), \ldots, m(b_n)) \\ &\iff R_{B_0}(b_1, \ldots, b_n) \\ &\implies R_{A_0}(g(b_1), \ldots, g(b_n)). \end{aligned}$$

Moreover, e is injective: for every pair $g(b), g(b') \in A_0$ we have

$$e(g(b)) = e(g(b')) \implies m(b) = m(b') \implies b = b' \implies g(b) = g(b').$$

Hence $e\colon A_0 \to B_0$ is an isomorphism in $\mathsf{Str}(\mathscr{S})$. This proves that e is c-reflexive.

($\Longleftarrow$) Suppose that e is c-reflexive, and let $h\colon X \to B$ be a $\mathsf{Str}(\mathscr{S})$-morphism with domain in $\mathscr{X}_c$, that is, $|X| < c$. Then $h[X] \subseteq B$ has cardinality $< c$. Hence, there exists a substructure $A_0 \subseteq A$ such that e restricts to an isomorphism $e\colon A_0 \xrightarrow{\cong} h[X]$. For every $x \in X$, let $g(x)$ be the unique element of A_0 such that $h(x) = e(g(x))$. This defines a function $g\colon X \to A$ satisfying $h = e \cdot g$. Moreover, g is a $\mathsf{Str}(\mathscr{S})$-morphism: for each n-ary relation symbol $R \in \mathscr{S}$ and $x_1, \ldots, x_n \in X$,

$$\begin{aligned} R_X(x_1,\ldots,x_n) &\implies R_B(h(x_1),\ldots,h(x_n)) \\ &\iff R_B(e(g(x_1)),\ldots,e(g(x_n))) \\ &\iff R_A(g(x_1),\ldots,g(x_n)). \end{aligned}$$

This proves that e lies in $\mathcal{E}_{\mathscr{X}_c}$. □

Corollary 4.8 *The data $\mathscr{A}$, $\mathscr{A}_0$, $(\mathcal{E},\mathcal{M})$, $\mathscr{X}$ satisfies the assumptions of Theorem 4.3.*

Proof. The category $\mathscr{A} = \mathbf{Alg}(\widehat{\Sigma})$ has products by Lemma 3.14, and its full subcategory $\mathscr{A}_0 = \mathbf{Alg}(\mathscr{C},\widehat{\Sigma})$ is closed under products and subalgebras by Lemma 3.5.

Moreover, $\mathscr{A} = \mathbf{Alg}(\widehat{\Sigma})$ is co-well-powered w.r.t. surjective morphisms: The collection of all $\widehat{\Sigma}$-algebras carried by a given set B forms a small set, and for every $\hat{\Sigma}$-algebra A and every quotient $e\colon A \twoheadrightarrow B$ in $\mathbf{Alg}(\widehat{\Sigma})$ one has $|B| \le |A|$, hence up to isomorphism there is only a small set of quotients of A.

Finally, every $\widehat{\Sigma}$-algebra A is an $\mathcal{E}_{\mathscr{X}}$-quotient (equivalently, a c-reflexive quotient) of some free algebra $T_{\widehat{\Sigma}}X$, where X is a c-clustered structure in $\mathsf{Str}(\mathscr{S})$. Indeed, let $m_j\colon X_j \rightarrowtail A$ ($j \in J$) be the family of all substructures of A such that $|X_j| < c$. Then $X = \coprod_{j\in J} X_j$ is c-clustered and the induced surjection $e = [m_j]_{j\in J}\colon X \twoheadrightarrow A$ is c-reflexive, as is its unique extension $e^{\#}\colon T_{\widehat{\Sigma}}X \twoheadrightarrow A$ to a $\widehat{\Sigma}$-algebra morphism. □

In the present setting, abstract equations in the sense of Definition 4.2(1) are surjective morphisms $e\colon T_{\widehat{\Sigma}}X \twoheadrightarrow E$ in $\mathbf{Alg}(\widehat{\Sigma})$ with codomain $E \in \mathbf{Alg}(\mathscr{C},\widehat{\Sigma})$, where X is a c-clustered structure. As we shall see in the proof of Theorem 4.16, they translate into the following concrete syntactic notion of equation:

Definition 4.9 A *c-clustered equation* over the set X of variables is an expression of either of the types

$$R_i(x_{i,1},\ldots,x_{i,n_i}) \ \ (i \in I) \ \ \vdash \ \ R(t_1,\ldots,t_n), \tag{4.2}$$

$$R_i(x_{i,1},\ldots,x_{i,n_i}) \ \ (i \in I) \ \ \vdash \ \ t_1 = t_2, \tag{4.3}$$

where (a) I is a set, (b) $x_{i,k} \in X$ for all i,k, (c) $t_1,\ldots,t_n$ are Σ-terms over X, (d) R_i ($i \in I$) and R are relation symbols in $\mathscr{S}$ with respective arities n_i and n, and (e) the set X can be expressed as a disjoint union $X = \coprod_{j\in J} X_j$ of subsets of cardinality $|X_j| < c$ such that for every $i \in I$, the variables $x_{i,1},\ldots,x_{i,n_i}$ all lie in the same set X_j. A c-clustered equation is *unconditional* if $I = \emptyset$.

Remark 4.10 (1) The key condition (e) restricts the level of connectedness of the variables. More formally, let the *Gaifman graph* of (4.2)/(4.3) be the undirected graph whose nodes are the variables in X and with an edge between $x, x' \in X$ iff x, x' both occur in $R_i(x_{i,1},\ldots,x_{i,n_i})$ for some $i \in I$. Condition (e) then expresses precisely that the connected components of the Gaifman graph all have cardinality $< c$.

(2) The above definition highlights an advantage of our categorical approach: the notion of c-clustered equation is guided by the fact that $\mathscr{X}$ consists of free algebras over c-clustered structures (and would arguably be rather hard to come up with ad hoc). The particular choice of $\mathscr{X}$ is in turn made to ensure that $\mathcal{E}_{\mathscr{X}}$ is rich enough to satisfy the categorical assumptions of Theorem 4.3; see also Example 4.19 below.

Definition 4.11 Let A be a $\widehat{\Sigma}$-algebra over $\mathscr{C}$.

(1) The algebra A *satisfies* the c-clustered equation (4.2) if for every map $h\colon X \to A$,

$$(R_i)_A(h(x_{i,1}),\ldots,h(x_{i,n_i})) \text{ for all } i \in I \qquad \text{implies} \qquad R_A(h^\sharp(t_1),\ldots,h^\sharp(t_n)).$$

Here $h^\sharp\colon T_\Sigma X \to A$ denotes the unique Σ-algebra morphism extending h.

(2) Similarly, A *satisfies* the c-clustered equation (4.3) if for every map $h\colon X \to A$,

$$(R_i)_A(h(x_{i,1}),\dots,h(x_{i,n_i})) \text{ for all } i \in I \qquad \text{implies} \qquad h^\sharp(t_1) = h^\sharp(t_2).$$

(3) A class of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ is a *c-variety* if it is axiomatizable by c-clustered equations.

Example 4.12 (Quantitative Algebras) For $\mathscr{C} = \mathsf{GMet}$ the c-clustered equations are of the form

$$x_i =_{\varepsilon_i} y_i \;\; (i \in I) \;\;\vdash\;\; t_1 =_\varepsilon t_2 \qquad \text{or} \qquad x_i =_{\varepsilon_i} y_i \;\; (i \in I) \;\;\vdash\;\; t_1 = t_2,$$

where (a) I is a set, (b) $x_i, y_i \in X$ for all $i \in I$, (c) t_1, t_2 are Σ-terms over X, (d) $\varepsilon_i, \varepsilon \in [0,1]$, and (e) the set X is a disjoint union $X = \coprod_{j\in J} X_j$ of subsets of cardinality $|X_j| < c$ such that for every $i \in I$, the variables x_i and y_i lie in the same set X_j. For ordinary metric spaces, these equations correspond to the c-clustered equations introduced by Milius and Urbat [18]. Concerning the special case $c = 2$, note that (i) 2-clustered equations can only contain trivial premises of the form $x =_\varepsilon x$, hence are equivalent to unconditional equations, and (ii) all quotients are 2-reflexive. Both (i) and (ii) are not true for generalized metric spaces if the axiom (Refl) is absent from $\mathsf{Ax}_{\mathrm{GM}}$, in which case $x =_\varepsilon x$ becomes a non-trivial condition.

Remark 4.13 In the case of metric spaces, c-clustered equations are closely related to the *c-basic equations* introduced by Mardare et al. [16], where condition (e) in Example 4.12 is replaced by the simpler (e') $|I| < c$. If c is an infinite regular cardinal, clearly every c-basic equation is c-clustered (with a single cluster). Conversely, if Σ is a κ-ary signature and $c \geq \kappa$, every c-clustered equation is equivalent to a c-basic equation [19, Rem. B.17]. However, for $c < \kappa$, c-clustered equations are more expressive than c-basic equations [2, App. A].

Example 4.14 (Ordered Algebras) For $\mathscr{C} = \mathsf{Pos}$ the c-clustered equations are of the form

$$x_i \leq y_i \;\; (i \in I) \;\;\vdash\;\; t_1 \leq t_2 \qquad \text{or} \qquad x_i \leq y_i \;\; (i \in I) \;\;\vdash\;\; t_1 = t_2,$$

subject to the conditions (a)–(c) and (e) as in Example 4.12.

Remark 4.15 The c-clustered equations for ordered algebras are related to *inequations in context* by Adámek et al. [4, Def. 3.15]. However, their notion of signature admit arities being finite posets, which allows to encode certain definedness constraints for operations using order relations on their arguments. If one restricts arities in their setting to finite discrete posets, then inequations in context essentially correspond to ω-clustered equations, where Σ is finitary. This is due to the fact that terms formed from operations with finite arity only contain finitely many variables, and so the index sets I may be chosen to be finite. Moreover, algebras in the sense of *op. cit.* are $\widehat{\Sigma}$-algebras in our sense where all arities of Σ are finite and where for each operation symbol one chooses the discrete lifting (Example 3.12(a)). Adámek et al. also consider *coherent* algebras, where every operation is monotone; restricting to finite discrete arities again, these algebras correspond to $\widehat{\Sigma}$-algebras where all arities are finite and where for each operation symbol Σ one chooses the product lifting (Example 3.12(b)). These $\widehat{\Sigma}$-algebras are the classical ordered algebras featuring in Bloom's variety theorem [8]. However, varieties in his setting are specified by unconditional inequations (without contexts); like in the metric case, these are equivalent to the 2-clustered equations.

With these preparations at hand, we establish our main result:

Theorem 4.16 (Variety Theorem) *A class of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ is a c-variety iff it is closed under c-reflexive quotients, subalgebras, and products.*

Proof. ($\Longrightarrow$) It suffices to show that for each c-clustered equation, the class of all $\widehat{\Sigma}$-algebras satisfying it has the required closure properties. We consider equations of type (4.2); the proof for (4.3) is analogous.

(1) *Closure under products.* Let $A = \prod_j A_j$ be a product of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ where each A_j satisfies (4.2), and suppose that $h\colon X \to A$ is a map such that $(R_i)_A(h(x_{i,1}), \ldots h(x_{i,n_i}))$ for all i. Denote by $\pi_j\colon A \to A_j$ the jth product projection. Then the map $h_j = \pi_j \cdot h$ satisfies $(R_i)_{A_j}(h_j(x_{i,1}), \ldots, h_j(x_{i,n_i}))$ for all i because π_j is relation-preserving. Since A_j satisfies (4.2) for all $j \in J$, it follows that $R_{A_j}(h_j^{\#}(t_1), \ldots, h_j^{\#}(t_n))$, which is equivalent to $R_{A_j}(\pi_j \cdot h^{\#}(t_1), \ldots, \pi_j \cdot h^{\#}(t_n))$ for all $j \in J$ and hence to $R_A(h^{\#}(t_1), \ldots, h^{\#}(t_n))$. This proves that A satisfies (4.2).

(2) *Closure under subalgebras.* Suppose that A is a $\widehat{\Sigma}$-algebra over $\mathscr{C}$ satisfying (4.2) and that $m\colon B \rightarrowtail A$ is a subalgebra. Let $h\colon X \to B$ be a map such that $(R_i)_B(h(x_{i,1}), \ldots, h(x_{i,n_i}))$ for all i. Then we have $(R_i)_A(m \cdot h(x_{i,1}), \ldots, m \cdot h(x_{i,n_i}))$ for all i because m is relation-preserving. Since A satisfies (4.2), it follows that $R_A((m \cdot h)^{\#}(t_1), \ldots, (m \cdot h)^{\#}(t_n))$, hence $R_A(m \cdot h^{\#}(t_1), \ldots, m \cdot h^{\#}(t_n))$, and thus $R_B(h^{\#}(t_1), \ldots, h^{\#}(t_n))$ because m is an embedding. This proves that B satisfies (4.2).

(3) *Closure under c-reflexive quotients.* Suppose that $e\colon A \twoheadrightarrow B$ is a c-reflexive quotient of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ and that A satisfies (4.2). Let $h\colon X \to B$ be a map such that $(R_i)_B(h(x_{i,1}), \ldots, h(x_{i,n_i}))$ for all i. Since (4.2) is c-clustered, we have $X = \coprod_{j \in J} X_j$ where $|X_j| < c$ for each j, and for each $i \in I$ the variables $x_{i,1}, \ldots, x_{i,n_i}$ lie in the same set X_j. Since $|h[X_j]| \le |X_j| < c$ and e is c-reflexive, the map e restricts to a $\mathsf{Str}(\mathscr{S})$-isomorphism $e\colon A_j \xrightarrow{\cong} h[X_j]$ for some $A_j \subseteq A$. For each $j \in J$ and $x \in X_j$, let $g(x)$ be the unique element of A_j such that $h(x) = e(g(x))$. This defines a map $g\colon X \to A$ such that $h = e \cdot g$. Using that the variables $x_{i,1}, \ldots, x_{i,n_i}$ lie in the same set X_j and $A_j \cong h[X_j]$, it follows from $(R_i)_B(h(x_{i,1}), \ldots, h(x_{i,n_i}))$ that $(R_i)_A(g(x_{i,1}), \ldots, g(x_{i,n_i}))$ holds. Therefore, since this holds for all $i \in I$ and A satisfies (4.2), we have $R_A(g^{\#}(t_1), \ldots, g^{\#}(t_n))$. Consequently, we have $R_B(e \cdot g^{\#}(t_1), \ldots, e \cdot g^{\#}(t_n))$, which in turn is equivalent to $R_B((e \cdot g)^{\#}(t_1), \ldots, (e \cdot g)^{\#}(t_n))$ and hence to $R_B(h^{\#}(t_1), \ldots, h^{\#}(t_n))$. This proves that B satisfies (4.2).

($\Longleftarrow$) We apply Theorem 4.3 to $\mathscr{A}$, $\mathscr{A}_0$, $(\mathcal{E}, \mathcal{M})$, $\mathscr{X}$ as chosen above (cf. Corollary 4.8). By the theorem, every class of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ closed under c-reflexive quotients, subalgebras, and products is axiomatizable by abstract equations $e\colon T_{\widehat{\Sigma}}X \twoheadrightarrow E$ where $E \in \mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$ and $X \in \mathsf{Str}(\mathscr{S})$ is c-clustered. Hence, it suffices to show that for every such e there exists an expressively equivalent set of c-clustered equations. We put

$$\Phi = \{\, R(x_1, \ldots, x_n) : R \in \mathscr{S},\, x_1, \ldots, x_n \in X \text{ and } R_X(x_1, \ldots, x_n) \,\}.$$

Since the structure X is c-clustered, there exist substructures $X_j \subseteq X$ ($j \in J$) of cardinality $|X_j| < c$ such that $X = \coprod_{j \in J} X_j$. From the description of the relations on the coproduct X (Remark 3.1(b)) we see that for every $R(x_1, \ldots, x_n)$ in Φ the variables $x_1, \ldots, x_n$ lie in the same component X_j. Using this we form the following set of c-clustered equations:

$$\begin{aligned} &\{\, \Phi \vdash R(t_1, \ldots, t_n) \;:\; R \in \mathscr{S},\, t_1, \ldots, t_n \in T_{\widehat{\Sigma}}X \text{ and } R_E(e(t_1), \ldots, e(t_n)) \,\} \\ \cup\; &\{\, \Phi \vdash t_1 = t_2 \;:\; t_1, t_2 \in T_{\widehat{\Sigma}}X \text{ and } e(t_1) = e(t_2) \,\}. \end{aligned} \tag{4.4}$$

We claim that e and (4.4) are expressively equivalent, that is, an algebra $A \in \mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$ satisfies the abstract equation $e\colon T_{\widehat{\Sigma}}X \twoheadrightarrow E$ iff it satisfies all the c-clustered equations in (4.4). Indeed, we have the following chain of equivalences, where the second step follows from the homomorphism theorem (Lemma 3.15):

$$\begin{aligned}
& A \text{ satisfies } e \\
\iff & \text{for all } h\colon X \to A \text{ in } \mathsf{Str}(\mathscr{S}), \text{ the map } h^\sharp\colon T_{\widehat{\Sigma}}X \to A \text{ factorizes through } e\colon T_{\widehat{\Sigma}}X \twoheadrightarrow E \\
\iff & \text{for all } h\colon X \to A \text{ in } \mathsf{Str}(\mathscr{S}), \\
& R_E(e(t_1),\dots e(t_n)) \text{ implies } R_A(h^\sharp(t_1),\dots h^\sharp(t_n)), \text{ for all } R \in \mathscr{S} \text{ and } t_1,\dots,t_n \in T_\Sigma X, \\
& \text{and } e(t_1) = e(t_2) \text{ implies } h^\sharp(t_1) = h^\sharp(t_2), \text{ for all } t_1, t_2 \in T_\Sigma X \\
\iff & \text{for all maps } h\colon X \to A \text{ such that} \\
& R_X(x_1,\dots,x_n) \text{ implies } R_A(h(x_1),\dots,h(x_n)), \text{ for all } R \in \mathscr{S} \text{ and } x_1,\dots,x_n \in X, \\
& R_E(e(t_1),\dots e(t_n)) \text{ implies } R_A(h^\sharp(t_1),\dots h^\sharp(t_n)), \text{ for all } R \in \mathscr{S} \text{ and } t_1,\dots,t_n \in T_\Sigma X, \\
& \text{and } e(t_1) = e(t_2) \text{ implies } h^\sharp(t_1) = h^\sharp(t_2), \text{ for all } t_1, t_2 \in T_\Sigma X \\
\iff & \text{for all maps } h\colon X \to A \text{ such that} \\
& R_A(h(x_1),\dots,h(x_n)) \text{ for all } R(x_1,\dots,x_n) \in \Phi, \\
& R_E(e(t_1),\dots e(t_n)) \text{ implies } R_A(h^\sharp(t_1),\dots h^\sharp(t_n)), \text{ for all } R \in \mathscr{S} \text{ and } t_1,\dots,t_n \in T_\Sigma X, \\
& \text{and } e(t_1) = e(t_2) \text{ implies } h^\sharp(t_1) = h^\sharp(t_2), \text{ for all } t_1, t_2 \in T_\Sigma X \\
\iff & A \text{ satisfies all the c-clustered equations in (4.4).} \qquad \square
\end{aligned}$$

As noted in Example 4.12, if $\mathscr{C}$ is the category of metric spaces, every quotient in $\mathscr{C}$ is 2-reflexive (that is, $\mathcal{E}_{\mathscr{X}} = \mathcal{E}$), and 2-clustered equations are equivalent to unconditional equations. This reasoning carries over to posets, but not to generalized metric spaces, or arbitrary relational structures. However, we can capture unconditional equations and varieties closed under all quotients via a different choice of $\mathscr{X}$:

$$\mathscr{X} \;=\; \text{all free algebras } T_{\widehat{\Sigma}}X \text{ where } X \in \mathsf{Str}(\mathscr{S}) \text{ is a discrete structure.}$$

Here, a structure X is *discrete* if $R_X = \emptyset$ for all $R \in \mathscr{S}$; hence discrete structures are essentially just sets. It is not difficult to verify that $\mathcal{E}_{\mathscr{X}} = \mathcal{E}$ and that abstract equations $e\colon T_{\widehat{\Sigma}}X \twoheadrightarrow E$ are expressively equivalent to unconditional equations; the reasoning is analogous to the proofs of Lemma 4.7 and Theorem 4.16. Consequently, we obtain as a further instance of the Abstract Variety Theorem:

Theorem 4.17 (Variety Theorem') *A class of $\widehat{\Sigma}$-algebras over $\mathscr{C}$ is axiomatizable by unconditional equations iff it is closed under quotients, subalgebras, and products.*

We conclude this section with a discussion of some applications of the variety theorem and of its relation to other approaches in the literature.

Example 4.18 (Ordered Algebras) For $\mathscr{C} = \mathsf{Pos}$, the cardinal number $c = 2$, and $\widehat{\Sigma}$ obtained by taking for every operation symbol the product lifting (Example 3.12(1b)), Theorem 4.16 yields Bloom's classical variety theorem [8] for ordered algebras. For all other choices of c and $\widehat{\Sigma}$, Theorem 4.16 instantiates to a family of new variety theorems for c-varieties of ordered algebras.

Example 4.19 (Quantitative Algebras) For $\mathscr{C}$ = metric spaces and again the product lifting for every operation symbol, Theorem 4.16 yields a refinement of the variety theorem by Mardare et al. [16]: a class of quantitative algebras is axiomatizable by c-clustered equations iff it is closed under c-reflexive quotients, subalgebras, and products.[10] For $\mathscr{C} = \mathsf{GMet}$ and arbitrary liftings, we obtain a family of new variety theorems for generalized quantitative algebras. Let us note that the interesting direction ($\Longleftarrow$) of our proof

[10] The variety theorem by Mardare et al. [16] works with c-basic equations (Remark 4.13) instead of c-clustered equations, but its statement is incorrect except for the cases where the two notions are equiexpressive; see [2, App. A] for a counterexample. Note that in our categorical setting, c-basic equations correspond to the choice $\mathscr{X}$ = free algebras $T_{\widehat{\Sigma}}X$ where $|X| < c$. The class $\mathcal{E}_{\mathscr{X}}$ then still consists of all c-reflexive quotients, but the key assumption of Theorem 4.3 is no longer satisfied: not every quantitative algebra is a c-reflexive quotient of an algebra in $\mathscr{X}$.

of Theorem 4.16, which proceeds by relating c-reflexive equations to abstract equations, is conceptually rather different from the proof of the quantitative variety theorem in *op. cit.*

Remark 4.20 Mardare et al. [16] also establish a *quasivariety theorem* for classes of finitary quantitative algebras axiomatizable by finitary conditional equations, i.e. expressions $s_i =_{\varepsilon_i} t_i \ \ (i \in I) \ \ \vdash \ \ s =_\varepsilon t$ where I is finite and s_i, t_i are arbitrary Σ-terms (rather than just variables). These classes are characterized by being closed under isomorphism, subalgebras, and subreduced products. Note that the scope of the quantitative quasivariety theorem is orthogonal to the quantitative variety theorem and its generalization to relational structures in the present paper: Unrestricted conditional equations are substantially more expressive than c-reflexive equations, while at the same time the quantitative quasivariety theorem neither applies to lifted signatures nor to infinitary operations and equations; in fact, since it is derived from the classical quasivariety theorem in model theory, an infinitary extension would be highly non-trivial.

Remark 4.21 In recent work, Mio et al. [21] develop an alternative approach to finitary quantitative universal algebra that avoids lifted signatures entirely. Their idea is to consider only quantitative algebras whose operations are arbitrary maps and express any desired nonexpansivity-type conditions via suitable quantitative equations (rather than via nonexpansivity w.r.t. a lifted signature). For instance, the assertion that an operation $\sigma_A \colon A^n \to A$ is α-Lipschitz, that is, nonexpansive w.r.t. the Lipschitz lifting $L_\sigma^{\mathsf{Lip},\alpha}$ of Example 3.12(a), may be expressed via the equations $x_i =_{\varepsilon/\alpha} x_i' \ (i < n) \ \vdash \ \sigma(x_1, \dots, x_n) =_\varepsilon \sigma(x_1', \dots, x_n')$ for $\varepsilon \in [0, 1]$. Similar equations can be given for the other liftings in Example 3.12. In fact, this approach applies to every finitary lifted signature due to the fact that every lifting of a finitary monad from Set to GMet admits a quantitative equational presentation [21, Thm 8.11]. We conjecture that this result easily extends from generalized metric spaces to arbitrary relational structures with infinitary Horn axioms.

In the context of variety theorems (which are not covered by Mio et al.), using lifted signatures has the advantage of providing an explicit separation between nonexpansivity-type conditions on the operations, and all the other axioms of a quantitative equational theory. This introduces more flexibility in the notion of variety. For example, non-expansivity of a c-ary operation can be expressed by a set of c^+-clustered equations (where c^+ is the successor cardinal of c), but not by c-clustered equations. Therefore, in the present setting involving lifted signatures there are c-varieties that would not be c-varieties otherwise.

Remark 4.22 Rosický [23] recently introduced *discrete enriched Lawvere theories* and developed a suitable notion of *clustered equation* at this level of categorical generality. The scope of the variety theorem of *op. cit.* is orthogonal to ours: It applies to algebras over locally presentable symmetric monoidal closed categories (which generalize our categories of relational structures), but unlike our present work it involves restrictions on the arities of operations and does not capture lifted signatures.

5 Exactness Property

It is well-known that for every Σ-algebra A, surjective Σ-algebra morphisms $e\colon A \twoheadrightarrow B$ are in bijective correspondence with congruence relations on A, which are equivalence relations respected by the operations $\sigma_A \colon A^n \to A$. Here we establish a corresponding exactness property for $\widehat{\Sigma}$-algebras, which turns out to be more involved and slightly subtle. For notational simplicity we assume in this section that the signature Σ is finitary; however, all statements easily extend to infinitary operations.

Recall from Notation 3.4 that $\mathscr{C} \hookrightarrow \mathsf{Str}(\mathscr{S})$ is the category of $\mathscr{S}$-structures satisfying the infinitary Horn clauses from Ax. Similarly, let $\mathscr{C}' \hookrightarrow \mathsf{Str}(\mathscr{S})$ denote the category of $\mathscr{S}$-structures satisfying the infinitary Horn clauses from Ax', the set of clauses of Ax of type (3.1) (that is, clauses of type (3.2) are dropped from Ax). For example, if $\mathscr{C}$ is the category of metric spaces, then $\mathscr{C}'$ is the category of pseudometric spaces because the axiom (Pos') of Example 3.6 is dropped. Note that $\mathscr{C} = \mathscr{C}'$ if Ax contains no axioms of type (3.2).

Definition 5.1 (1) Given a $\widehat{\Sigma}$-algebra A over $\mathscr{C}$ with underlying $\mathscr{S}$-structure $(A, (R_A)_{R\in\mathscr{S}})$, a *refining structure* on A is an $\mathscr{S}$-structure $(R_A')_{R\in\mathscr{S}}$ with carrier A satisfying the following properties:

(a) $(A, (R_A')_{R\in\mathscr{S}})$ lies in $\mathscr{C}'$;

(b) $R_A \subseteq R'_A$ for each $R \in \mathscr{S}$;

(c) for each $\sigma \in \Sigma$, the operation σ_A is relation-preserving w.r.t. the relations R'_A and $L_\sigma(R'_A)$:

$$L_\sigma(R'_A)((a_{i,1})_{i<n}, \dots, (a_{i,m})_{i<n}) \implies R'_A(\sigma_A((a_{i,1})_{i<n}), \dots, \sigma_A((a_{i,m})_{i<n})) \quad \text{for every } R \in \mathscr{S},$$

where n is a the arity of σ, m is the arity of R, and $a_{i,j} \in A$.

(2) A *congruence* on A is an equivalence relation $\equiv$ on A such that, for all $\sigma \in \Sigma$ of arity n and all $a_i, a'_i \in A$, $i = 1, \dots, n$, we have

$$a_i \equiv a'_i \ \ (i < n) \implies \sigma_A(a_1, \dots, a_n) \equiv \sigma_A(a'_1, \dots, a'_n).$$

(3) A *compatible pair* $((R'_A)_{R\in\mathscr{S}}, \equiv)$ on A is given by a refining structure $(R'_A)_{R\in\mathscr{S}}$ and a congruence $\equiv$ on A satisfying the following conditions:

(a) For each n-ary $R \in \mathscr{S}$ and $a_i, a'_i \in A$, $i = 1, \dots, n$, we have

$$a_i \equiv a'_i \ \ (i < n) \implies \big(R'_A(a_1, \dots, a_n) \iff R'_A(a'_1, \dots, a'_n)\big).$$

(b) For all axioms of type (3.2) in Ax and $h\colon X \to A$,

$$(R'_i)_A(h(x_{i,1}), \dots, h(x_{i,n_i})) \text{ for all } i \in I \qquad \text{implies} \qquad h(x_1) \equiv h(x_2).$$

Example 5.2 (Quantitative Algebras) For $\mathscr{C} = \mathsf{GMet}$, Definition 5.1 can be rephrased as follows. A *generalized pseudometric* is a fuzzy relation satisfying all axioms from $\mathsf{Ax}_{\mathrm{GM}}$ except possibly (Pos). We assume that each lifting L_σ restricts to an endofunctor $L_\sigma\colon \mathsf{GPMet} \to \mathsf{GPMet}$ on the category of generalized pseudometric spaces and nonexpansive maps. For each $(A, p) \in \mathsf{GPMet}$ we write $L_\sigma(p)$ for the generalized pseudometric on $L_\sigma(A, p)$.

(1) Given a $\widehat{\Sigma}$-algebra (A, d_A) over GMet, a fuzzy relation p on A is a *refining generalized pseudometric* if

(a) p is a generalized pseudometric,

(b) $p(a, a') \le d_A(a, a')$ for all $a, a' \in A$, and

(c) for each $\sigma \in \Sigma$, the operation σ_A is nonexpansive w.r.t. p and $L_\sigma(p)$:

$$p(\sigma_A((a_i)_{i<n}), \sigma_A((b_i)_{i<n})) \le L_\sigma(p)((a_i)_{i<n}, (b_i)_{i<n}).$$

(2) A refining generalized pseudometric p and a congruence $\equiv$ are *compatible* if

$$a \equiv b \wedge a' \equiv b' \implies p(a, a') = p(b, b')$$

for all $a, a', b, b' \in A$, and furthermore, if $\mathsf{Ax}_{\mathrm{GM}}$ contains (Pos), then

$$p(a, a') = 0 \implies a \equiv a'.$$

Then a compatible pair $(p, \equiv)$ corresponds to a compatible pair $(((R'_\varepsilon)_A)_{\varepsilon\in[0,1]}, \equiv)$; the correspondence between p and $((R'_\varepsilon)_A)_{\varepsilon\in[0,1]}$ is obtained as in Example 3.6.

Now for each $A \in \mathbf{Alg}(\widehat{\Sigma})$ there is an order on compatible pairs on A defined by

$$((R'_A)_{R\in\mathscr{S}}, \equiv) \le ((R''_A)_{R\in\mathscr{S}}, \equiv') \qquad \text{iff} \qquad R'_A \subseteq R''_A \text{ for all } R \in \mathscr{S} \text{ and } {\equiv} \subseteq {\equiv'}.$$

Similarly, quotients of A are ordered by $e \le e'$ iff $e' = h \cdot e$ for some $\widehat{\Sigma}$-algebra morphism h. A *$\mathscr{C}$-quotient* is a quotient with codomain in $\mathbf{Alg}(\mathscr{C}, \widehat{\Sigma})$. Under the above orders, both compatible pairs and $\mathscr{C}$-quotients of A form complete lattices.

We let $\mathcal{E}_\leftrightarrow$ denote the class of all quotients in $\mathsf{Str}(\mathscr{S})$ that both preserve and reflect relations.

Theorem 5.3 (Exactness) *Suppose that $\hat{\Sigma}$ is a lifted signature satisfying $L_\sigma(\mathcal{E}_\leftrightarrow) \subseteq \mathcal{E}_\leftrightarrow$ for all $\sigma \in \Sigma$. Then for $A \in \mathbf{Alg}(\widehat{\Sigma})$ the complete lattices of $\mathscr{C}$-quotients of A and compatible pairs on A are isomorphic.*

Hence, for free algebras $A = T_{\widehat{\Sigma}}X$ the theorem fully characterizes abstract equations $e\colon T_{\widehat{\Sigma}}X \twoheadrightarrow E$.

Proof sketch. For every $\mathscr{C}$-quotient $e\colon A \twoheadrightarrow B$ we obtain the compatible pair $((R_e)_{R\in\mathscr{S}}, \equiv_e)$ defined by

$$R_e(a_1,\dots,a_n) \iff R_B(e(a_1),\dots,e(a_n)) \qquad \text{and} \qquad a \equiv_e a' \iff e(a) = e(a')$$

for each n-ary $R \in \mathscr{S}$ and $a_1,\dots,a_n,a,a' \in A$. In the reverse direction, every compatible pair $P = ((R'_A)_{R\in\mathscr{S}}, \equiv)$ yields a $\mathscr{C}$-quotient by forming the quotient $\widehat{\Sigma}$-algebra $e_P\colon A \twoheadrightarrow A/{\equiv}$, where $A/{\equiv}$ is the quotient Σ-algebra induced by the congruence $\equiv$ with relations defined by

$$R_{A/\equiv}([a_1],\dots,[a_n]) \iff R'_A(a_1,\dots,a_n)$$

for each n-ary $R \in \mathscr{S}$ and $a_1,\dots,a_n \in A$. It is not difficult to verify that the two maps

$$e \;\mapsto\; ((R_e)_{R\in\mathscr{S}}, \equiv_e) \qquad \text{and} \qquad P = ((R'_A)_{R\in\mathscr{S}}, \equiv) \;\mapsto\; e_P$$

are monotone and mutually inverse. □

6 Conclusions and Future Work

We have investigated clustered algebraic equations over relational structures, which generalizes and unifies a number of related notions that naturally appear in algebraic reasoning over metric spaces or posets. Our key insight is that this notion is actually an instance of abstract morphic equations in a categorical framework and that the characterization of its expressive power can be presented as an instance of an abstract Birkhoff-type variety theorem. Apart from simplifying proofs, the generality of the categorical approach highlights the clear separation between algebraic and relational aspects in equational reasoning, which often remains implicit when algebras over specific structures (such as metric spaces) are considered.

A natural next step is to derive a complete deduction system for equations with relational features. This should be achievable in a systematic manner much like in our approach to the variety theorem by combining the abstract completeness theorem by Milius and Urbat [18, Thm. 4.4] with our exactness theorem for relational algebras (Theorem 5.3). We expect a tight connection to existing completeness results for generalized quantitative algebras by Mio et al. [20, 21] and for algebras over infinitary Horn structures by Ford et al. [10].

A further direction is to relate our work to the recent investigation of monads over metric spaces [2,6,22] and posets [3, 4]. In all these works, notions of equational theories are characterized by properties of their corresponding free-algebra monads. In our setting, characterizing the monads on the category $\mathscr{C}$ corresponding to c-varieties remains an open problem, which we expect to be quite challenging in general.

Another potential direction is to investigate whether clustered algebraic equations over relational structures can be expressed alternatively via Lawvere theories with arities [17].

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