

Algebra of Self-Expression

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Abstract

Typical arguments for results like Kleene’s Second Recursion Theorem and the existence of self-writing computer programs bear the fingerprints of equational reasoning and combinatory logic. In fact, the connection of combinatory logic and computability theory is very old, and this paper extends this connection in new ways. In one direction, we counter the main trend in both computability theory and combinatory logic of heading straight to undecidability. Instead, this paper proposes using several very small equational logics to examine results in computability theory itself. These logics are decidable via term rewriting. We argue that they have something interesting to say about computability theory. They are closely related to fragments of combinatory logic which are decidable, and so this paper contributes to the study of such fragments. The paper has a few surprising results such as a classification of quine programs (programs which output themselves) in two decidable fragments. The classification goes via examination of normal forms in term rewriting systems, hence the title of the paper. The classification is an explanation of why all quine programs (in any language) are “pretty much the same, except for inessential details.” In addition, we study the relational structure whose objects are the programs with the relation $p \rightarrow q$ (read “ p expresses q ”) if program p when run on nothing outputs q .

Keywords: Kleene’s Recursion Theorem, quine programs, combinatory logic, term rewriting, decidable equational logic

1 Introduction

The modest point made in this paper is that several results in computability theory call on equational reasoning of a very simple sort. We are thinking of the equations seen in the s_n^m -Theorem or the universal program, and of even more interest here are equations pertaining to a “diagonal operator” that one can see in proofs of Kleene’s Second Recursion Theorem. We formulate several systems of equations (as in abstract data types or equational logic) which can then be studied using term rewriting theory. The particular systems of equations which we study turn out to be terminating and confluent, and thus the initial algebras are computable. They are also quite close to fragments of combinatory logic, but the fact that they have easily-described computable models means that they are much weaker than combinatory logic as one usually sees it. This loss of expressive power has a compensation: we can study the normal forms in our rewriting systems and prove a few results which are unexpected, such as a characterization of quine programs in the language of the diagonal operator and program sequencing.

1.1 Kleene’s Second Recursion Theorem: two proofs

Let us begin with one of the cornerstone results in computability theory, Kleene’s Second Recursion Theorem [6]. This result is more often called *the Recursion Theorem*. We are not interested in its many

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applications. (For some of those, especially in classical recursion theory and descriptive set theory, see Moschovakis [9]. For other applications, this time in connection with computer viruses, see Bonfante et al [2].) Instead, we are only interested here in the proof. We are going to formulate this in terms of *programs* in some unspecified language, whereas the classical result uses natural numbers as *indices* of *computable functions*. In the classical setting, a number e is an index of an n -place partial function $\varphi_e^n(x_1, \dots, x_n)$. In our setting, or in any setting involving nested use of this notation, it is more readable to write this as $\llbracket e \rrbracket^n(x_1, \dots, x_n)$. Indeed, we also usually drop the superscript n ; it will always be clear from the context. We are going to state the Recursion Theorem for $n = 2$ for simplicity.

Theorem 1.1 *Let p be a program, consider as a function $\llbracket p \rrbracket$ of two arguments. There is a program q^* so that for all r ,*

$$\llbracket q^* \rrbracket(r) \simeq \llbracket p \rrbracket(q^*, r).$$

Proof. Here is the original proof due to Kleene. Recall the program s_1^1 ; it has the property that

$$\llbracket \llbracket s_1^1 \rrbracket(x, y) \rrbracket(z) = \llbracket x \rrbracket(y, z). \quad (1)$$

Fix p and then modify it to get p' with the property that for all x, r ,

$$\llbracket p' \rrbracket(x, r) = \llbracket p \rrbracket(\llbracket s_1^1 \rrbracket(x, x), r).$$

Then take q^* to be $\llbracket s_1^1 \rrbracket(p', p')$. To see that q^* defined this way has the property that we want, we calculate:

$$\llbracket q^* \rrbracket(r) = \llbracket \llbracket s_1^1 \rrbracket(p', p') \rrbracket(r) = \llbracket p' \rrbracket(p', r) = \llbracket p \rrbracket(\llbracket s_1^1 \rrbracket(p', p'), r) = \llbracket p \rrbracket(q^*, r).$$

Here is a second proof, from [10]. Modify p to get \hat{q} so that for all x and r :

$$\llbracket \llbracket \hat{q} \rrbracket(x) \rrbracket(r) \simeq \llbracket p \rrbracket(\llbracket x \rrbracket(x), r).$$

Then we can take x to be \hat{q} itself, and we would have that for all r ,

$$\llbracket \llbracket \hat{q} \rrbracket(\hat{q}) \rrbracket(r) \simeq \llbracket p \rrbracket(\llbracket \hat{q} \rrbracket(\hat{q}), r).$$

Thus when we take q^* to be $\llbracket \hat{q} \rrbracket(\hat{q})$, we would have $\llbracket q^* \rrbracket(r) \simeq \llbracket p \rrbracket(q^*, r)$. \square

There are several questions which we want to ask about the two proofs. Are they the same? If not, what is common to them? Why should anyone care?

The two proofs are of course not the same. This matter was studied by Jones [5] in connection with in-practice computational systems that make use of the Recursion Theorem. It turns out that the two different constructions are correlated with performance differences in a certain algorithm mentioned in [5]. Be that as it may, there is evidently something similar about the two proofs. They both employ what we called “modification”. This is a very general technique and we shall see a version of it under the name of *combinatory completeness*. In the original proof, Kleene is using (1) and a bit of *equational reasoning*.

This paper is not about the Recursion Theorem per se but about equational reasoning in both of these proofs. Indeed, we are interested in equational logics which are “small” enough to be decidable and yet “big” enough to say something interesting about fundamental results in computability theory. We ultimately would like to return to the Recursion Theorem and if possible to classify its equational proofs. What we have done in this paper is to take a step towards this goal by formulating an equational logic related to the diagonalization operation $p \mapsto \llbracket p \rrbracket(p)$ that we see in the second proof of the Recursion Theorem. Our main results concern two equational systems related to this operation. The reason why we are not yet able to formulate a logic that covers the first proof as well comes from the fact that this proof is more about substitution than about diagonalization. To make this point clearer, we digress to study quine programs.

1.2 Some problems about program “expression”

A *quine program* is a program p such that when p is run “without input”, the output is p . We have drawn attention to the “without input” point. To formalize this as we have done above, we would say that $\llbracket p \rrbracket^0() \simeq p$. When we think of programs via natural number indices, this would seem to be a natural move. However, when we think of them as programs which operate on (k -tuples of) words for some k , we could equally well say a quine program is a program p such that $\llbracket p \rrbracket^1(\varepsilon) \simeq p$, where ε is the empty word.

Perhaps the easiest way to construct such a program is to use the Recursion Theorem, but not exactly in the form which we saw in Theorem 1.1. We want a slightly different version: Let p be a program, and consider the function $\llbracket p \rrbracket^1$. There is a program q^* so that $\llbracket q^* \rrbracket() \simeq \llbracket p \rrbracket(q^*)$. Then we would take p to be any program such that $\llbracket p \rrbracket$ is the identity function. For this p , the promised program q^* is a quine program.

One classical construction of a quine program in a sense bypasses the much more general (and thus better) construction offered by the Recursion Theorem. It can be summarized as follows. Let $diag$ be a program such that for all x , $\llbracket \llbracket diag \rrbracket(x) \rrbracket() \simeq \llbracket x \rrbracket(x)$. Then $\llbracket \llbracket diag \rrbracket(diag) \rrbracket() \simeq \llbracket diag \rrbracket(diag)$. So $\llbracket diag \rrbracket(diag)$ is a quine program. For an example, here is the quine in Lisp:

```
((lambda (x) (list x (list 'quote x))) '(lambda (x) (list x (list 'quote x))))
```

So here $diag$ would be the program `((lambda (x) (list x (list 'quote x))))`. However, in other languages, the most direct quine program would not be obtained in this way. For example, looking at quines in Python on the internet, I have not seen any that seem to come from a $diag$ program in this way. Instead, they rely on string substitution. I have in mind examples like

```
var = "print('var = ', repr(var), '\\neval(var)')"  
eval(var)
```

Getting an account of all the possible quine programs using substitution is beyond the scope of this paper. What we do is to construct an equational logic related to the diagonalization operator and prove results about that.

1.3 Program expression

The quine program may be put into a larger context. Marvin Minsky wrote “It is generally recognized that the greatest advances in modern computers came through the notion that programs could be kept in the same memory with ‘data,’ and that programs could operate on other programs, or on themselves, as though they were data.” Let us take seriously the idea that programs can *output* other programs as well. For programs p and q , write

$$p \rightarrow q \quad \text{iff} \quad \llbracket p \rrbracket() \simeq q \tag{2}$$

We read this as p “expresses” q . That is, running p on “nothing” outputs q . If we take seriously Minsky’s point, it would seem that the study of the graph (*Programs*, \rightarrow) would be of interest in computer science. A quine program is one which expresses itself. Are there pairs of different programs which express each other, or about cycles of other lengths? Are there infinite sequences $p_0 \rightarrow p_1 \rightarrow p_2 \rightarrow \dots$ of distinct programs? The answers to both of these questions is Yes. There are cycles of all lengths and infinite sequences. The proofs (at least the only proofs which I knew before working on this topic) used the Recursion Theorem, or something close. They were not “equational.” So the purpose of this paper is to ask about equational proofs of the main facts of the program-expression graph.

1.4 Plan for the paper

We are concerned with the equational reasoning that goes on in the basic parts of computability. So the main proposal in the paper is to formulate an equational logic whose operations include sequencing of instructions (written with the symbol $+$) as well as application of one program to another (written

$x + e = x$	$d @ x = (w @ x) + x$	$w @ e = e$
$e + x = x$	$e @ x = x$	$s_1^1 @ x @ y @ z = x @ y @ z$
$(x + y) + z = x + (y + z)$	$w @(x + y) = (w @ x) + (w @ y)$	$u_0 @ x = x @ e$
$(x + y) @ z = y @(x @ z)$	$w @ x @ y = y + x$	$u_1 @ x @ y = x @ y$

Fig. 1. The full set E of equations used in this paper.

@, and sometimes elided). The constants are the empty program e , the program s_1^1 , and the universal programs u_0 and u_1 . In addition, we are particularly interested in two constants which are not part of the standard presentations of computability theory. One is d , related to the diagonal map $x \mapsto \llbracket x \rrbracket(x)$. (But our intended interpretation of d is not quite this map.) The other is w , intended to be $x \mapsto$ “the instructions to write x ”. The set of equations pertinent to all of this is in Figure 1.

Our plan is to study this equational logic in fragments. Following a short look at the background in Section 2, we begin with one of the two main fragments in the paper: the one with e , d , $+$ and $@$; this is studied in Section 3. In Section 4, we add w . There is no section devoted only to e , w (rather than d), $+$ and $@$: although this fragment is natural to consider, it is too weak for the purposes of this paper. We have a few minor remarks about the larger fragments and some other topics in Section 5. Two discussions have been moved to the Appendix.

2 Background

The background needed to read this paper is very minimal. The main required notions are those from equational logic and term rewriting theory. We review this shortly. It would be good to have seen combinatory logic (CL), but this is mainly to compare what we are doing to CL. Except for that material, the paper should be readable even for someone who has never seen CL.

2.1 Equational logic, term rewriting, and computable algebras

Let E be a set of equations. We write $t \equiv u$ if $E \vdash t = u$ using E and the inference rules of equational logic. These rules are the reflexive, symmetric, and transitive properties of equality, substitution, and congruence for all function symbols. We assume familiarity with the definitions and basic results mentioned just above. Here are the results which we shall be using.

Theorem 2.1 ([8], **Theorem 51**) *Suppose that E is a finite set of equations which can be oriented to give a term rewrite system which is terminating and confluent. Then every term t has a unique normal form $\text{nf}(t)$. Moreover, we have*

$$t \equiv u \quad \text{iff} \quad \text{nf}(t) = \text{nf}(u).$$

The set of normal forms is the initial algebra of E with a canonical structure:

$$f(\text{nf}(t_1), \dots, \text{nf}(t_n)) = \text{nf}(f(t_1, \dots, t_n)).$$

This algebra is a computable algebra.

The assumption in this last result that E can be oriented to give a terminating and confluent term rewriting system is rather strong. In general, it fails.

Theorem 2.2 (Perkins [11]) *The question, given a finite set E of equations and an equation $t = u$, of whether or not $t \equiv_E u$, is undecidable.*

One concrete setting where this applies is to combinatory logic, reviewed just below.

2.2 Combinatory algebras

For background on combinatory logic, see [1].

Definition 2.3 A combinatory algebra (CA) is a tuple $(D, \cdot, \mathbf{s}, \mathbf{k})$ such that D is a set, \cdot is a (total) binary operation on D , \mathbf{s} and \mathbf{k} are elements of D , and

$$\begin{aligned}\mathbf{k} \cdot x \cdot y &= x \\ \mathbf{s} \cdot x \cdot y \cdot z &= x \cdot z \cdot (y \cdot z)\end{aligned}$$

In the equations above, we followed the standard notation of omitting parentheses. For example $\mathbf{s} \cdot x \cdot y \cdot z$ is really $((\mathbf{s} \cdot x) \cdot y) \cdot z$, and $x \cdot z \cdot (y \cdot z)$ is really $(x \cdot z) \cdot (y \cdot z)$. In addition, one usually omits the \cdot entirely and writes, for example, $\mathbf{s}xyz = xz(yz)$.

Theorem 2.4 (Combinatory Completeness) For every term $t(x_1, \dots, x_n)$ in variables x_1, x_2, \dots, x_n there is an element $t^* \in D$ such that $t^* \cdot d_1 \cdot d_2 \cdots d_n = t(d_1, \dots, d_n)$ for all $d_1, \dots, d_n \in D$.

Example 2.5 With t the term $x(yz)$, we can take $\mathbf{b} = \mathbf{s} \cdot (\mathbf{k} \cdot \mathbf{s}) \cdot \mathbf{k}$. This is because

$$\mathbf{s}(\mathbf{k}\mathbf{s})\mathbf{k}xyz = (\mathbf{k}\mathbf{s})x(\mathbf{k}y)z = \mathbf{s}(\mathbf{k}x)yz = \mathbf{k}xz(yz) = x(yz).$$

Another example: let t be the term $y(x(z))$. Let $\mathbf{c} = \mathbf{s}(\mathbf{b}\mathbf{b}\mathbf{s})(\mathbf{k}\mathbf{k})$, and let $\mathbf{b}' = \mathbf{c}\mathbf{b}$. Then one can show that \mathbf{b}' has the required property: $\mathbf{b}'xyz = y(xz)$. The details of why this works are not important here; the main thing is the combinatory completeness result itself.

Here is an outline of the general method for *solving fixed point equations* in CL, using combinatory completeness. Consider a function $f(x)$ such as $f(x) = x \cdot e$. We show that there is a term x^* so that $f(x^*) \equiv x^*$. For this, consider the term $f(x \cdot x)$. By combinatory completeness, let t^* be such that for all x , $t^* \cdot x \equiv f(x \cdot x)$. Let $x^* = t^* \cdot t^*$. Then

$$x^* = t^* \cdot t^* \equiv f(t^* \cdot t^*) = f(x^*).$$

The second proof of Kleene's Second Recursion Theorem is basically an application of this general method. In our second proof of Theorem 1.1, one starts with $\llbracket p \rrbracket(x, r)$ and then considers $\llbracket p \rrbracket(\llbracket x \rrbracket(x)r)$. Now $\llbracket x \rrbracket(x)$ is quite closely related to $x \cdot x$. Indeed, one would like to take the natural numbers (indices of partial computable functions) and then construct a combinatory algebra using $x \cdot y = \varphi_x(y) = \llbracket x \rrbracket(y)$ together with chosen indices to interpret \mathbf{s} and \mathbf{k} . The problem with this is that combinatory algebras are structures in the sense of universal algebra and thus must interpret their symbols by total functions. So this idea does not work out – but it shows the connection between fixed points in CL and this proof of the Recursion Theorem.

Here are some further comments related to this last point. First, this issue of total and partial interpretations will return in this paper in interesting ways. Second, we mention other work related to combinatory logic which and category theory which might well be connected to our subject. These are the recent paper by Roberts [12] on general settings for diagonalization arguments, and the older paper by Di Paola and Heller [3] which offers a category-theoretic treatment of partiality in computability theory.

2.3 Related work

To the best of our knowledge, there are no studies of the equational systems proposed in this paper. There are decidable fragments of combinatory logic; for example, see [14, 16]. Our proposal is (with minor differences) another decidable fragment of CL. However, our work is different for a number of reasons. First and foremost, our motivation comes from the equational analysis of basic results in computability, not from CL itself. Second, the particular equational systems here are different from what we have seen in the literature, and even small differences in equational systems sometimes result in very different technical developments.

$x + \mathbf{e} = x$	$(\mathbf{d} @ x) @ y = x @ (y + x)$
$\mathbf{e} + x = x$	$\mathbf{e} @ x = x$
$(x + y) + z = x + (y + z)$	$\mathbf{d} @ \mathbf{e} = \mathbf{e}$
$(x + y) @ z = y @ (x @ z)$	

Fig. 2. The set E of equations for the language studied in Section 3. The signature has constants \mathbf{d} and \mathbf{e} , and function symbols $+$ and $@$.

3 Equational logic of diagonalization

We present an equational logic in the style of combinatory logic (Section 2.2). We take constants \mathbf{d} and \mathbf{e} , and also two binary function symbols, $@$ (for application) and $+$ (for concatenation). In the language of combinatory logic, $t @ u$ is just a variant symbol for $t \cdot u$ (usually written tu) and $t + u$ is a variant notation for $\mathbf{b} \cdot t \cdot u$. We are interested in the set E of equations shown in Figure 2. In later sections, we add constant symbols for other basic operations in computability theory and also add equations to E .

We speak of $@$ -terms and $+$ -terms, and these are defined in the evident way by looking at the top-most symbol. For example $\mathbf{d} @ (\mathbf{e} + \mathbf{d})$ is an $@$ -term, and $(\mathbf{e} @ \mathbf{d}) + \mathbf{d}$ is a $+$ -term.

3.1 The idea behind the equations, in plain terms

Recall the quote from Minsky emphasizing how important it was to see that “programs could operate on other programs, or on themselves, as though they were data.” Let us think of $@$ as application of one program to another, and $+$ as sequentialization, and let us restrict attention to languages \mathcal{L} where $@$ and $+$ are first-class constructs. At first glance one might expect the equations in this paper to hold for \mathcal{L} . But this cannot literally be right, since the application construct is partial. Replacing $=$ by the weaker \simeq , we would expect the equations to all hold. However, we want to work in standard equational logic and so we want to think about total interpretations. Thus, we would like an interpretation of the initial algebra of our equations into \mathcal{L}

$$*: T_{\Sigma}/E \rightarrow \mathcal{L}$$

We are not going to exhibit this map $*$ for any real programming languages (but see our earlier example of a quine in Lisp). Instead, we look to a stylized version of a programming language. So we are looking at pseudocode and showing how one can interpret the symbols $+$, $@$, \mathbf{d} , and \mathbf{e} in such a way as to satisfy the equations.

Let us consider a set P of “programs” written in English.³ These are sequences of *instructions*. We only want instructions of a very simple form, including basic imperative verbs like *sit* and *stand*. These basic programs should be intransitive verbs (no object). For a transitive verb like *write*, we want the ability to take an input word w (in the “register”, see below) and replaces the register contents with a program which would write w .

Let us spell out this idea in a little more detail, in effect defining P (somewhat informally) and also giving its operational semantics.

For example, P should contain the program *sit*. Executing this program causes the person doing so to sit. P also should contain the program *write “sit”*. Executing this program causes the person doing so to write the word “sit”. This writing should go on a *register*. We might think of this register as a blank page to start. But even if the register were not blank, it would make sense to write the word “sit” into the register. For this, we write onto the end of the register. This would be the opposite end from what we read from. P should be closed under concatenation. For this, let us use a comma. So P should have a program *stand, write “sit”*.

³ Readers unhappy with this may omit this section and jump to Section 3.3. Very little in the rest of this paper depends on the interpretation of our terms into programs in language P .

Now we can endow P with the structure of a Σ -algebra which satisfies the equations in Figure 2. We interpret \mathbf{e} by the empty program, and we take $+$ to be the operation taking programs p and q to be “ p, q ” when p and q are non-empty, and otherwise is just the ordinary concatenation of p and q . Notice the comma here in this first case. Then the first three equations in Figure 2 hold.

We add the comma mainly because it looks better to do so. But we could resort to other devices: we could start instructions with upper-case letters, for example.

Let $\mathbf{d}^* = \textit{diag}$ and $\mathbf{w}^* = \textit{write}$, the programs given below:

$$\begin{array}{ll} \textit{write} : & \textit{write the instructions to write what is in the register} \\ \textit{diag} : & \textit{write the instructions to write what is in the register in front of it} \end{array} \quad (3)$$

For example, executing the program *write* when *diag* is in the register results in the register having

write “*write the instructions to write what is in the register in front of it*”

Executing the program *diag* when the register contains the program *sit* results in the register having the program *write* “*sit*”, *sit*. Then executing this last program with a word q in the register would cause the one doing so to write “*sit*” at the end of the register (so the register will contain q, \textit{sit}), and then to sit.

Further, let us run *diag* when the register contains *stand, sit*. We would get

write “*stand, sit*”, *stand, sit*

If we now run this program when the register contains q , then the register would contain $q, \textit{stand, sit}$ and the executor would stand and then sit.

Let $(t @ u)^*$ be the result of running t^* with u^* in the register. (We also describe this by saying that we apply t^* to u^* .) That is, we interpret the binary function symbol $@$ by the function $\textit{app} : P^2 \rightarrow P$ given by

$$\textit{app}(p, q) = \textit{the program in the register after we apply } p \textit{ to } q$$

With a sufficiently rich base language, this operation will be (merely) partial. We thus are not claiming that our language P is itself a model of the equations under consideration. We do claim that the interpretation function \textit{app} is total on the image of $*$. We postpone the discussion of this to Section 3.5.

One of our equations is $(x + y) @ z = y @ (x @ z)$. Suppose that x is *write* “*die*”, y is *write* “*hop*”, and z is the empty word. Then $x + y$ is *write* “*die, hop*”, and $(x + y) @ z$ would give *die, hop* in the register. On the other side, $x @ z$ would give *die* in the register, and so executing *write* “*hop*” on this would give *die, hop*.

Another of our equations is $(\mathbf{d} @ x) @ y = x @ (y + x)$. As a way of verifying it by example, take x to be *hop* and y to be *stand*. Then $\mathbf{d} @ x$ is *write* “*hop*”, *hop*. Executing this with the register containing y would give *stand, hop* in the register, and the executor hops. This is the interpretation of $(\mathbf{d} @ x) @ y$. On the other side, $y + x$ is *stand, hop*, and executing x when this is in the register does not change the register, but the executor hops.

The last two equations are $\mathbf{e} @ x = x$ and $\mathbf{d} @ \mathbf{e} = \mathbf{e}$. Since we take \mathbf{e} to be the empty program, these seem sensible. But if we take \mathbf{e} to be some other “do nothing” program (think of *skip*), then it might well be the case that $\mathbf{d} @ \mathbf{e} \neq \mathbf{e}$. Indeed, this last equation $\mathbf{d} @ \mathbf{e} = \mathbf{e}$ might not hold “on the nose.” It might hold “up to extensionality”. That is, if we add a rule of extensionality to equational logic, inferring $x = y$ from $xz = yz$, then $\mathbf{d} @ \mathbf{e} = \mathbf{e}$ follows as a consequence.

3.2 The quine program in this example

Recall that we are defining a function $*$: $T_\Sigma/E \rightarrow P$. We shall see in Example 3.3 that $(\mathbf{d} @ \mathbf{d}) @ \mathbf{e} = \mathbf{d} @ \mathbf{d}$. This suggests that $(\mathbf{d} @ \mathbf{d})^*$ is a quine program: when run on an empty register, it “expresses” (outputs in the register) itself. To check this, note that $(\mathbf{d} @ \mathbf{d})^*$ is

write “*write the instructions to write what you see in front of it*”, *write the instructions to write what you see in front of it*

Then the reader can check that executing this program on an empty register outputs itself.

3.2.1 A comment on the equations

We *don't* want the *directly self-referential* program “*print this sentence*” because the issue under discussion is whether we can achieve the effect of this, but with much less powerful means.

3.3 Rewriting system and normal forms

A *rewrite rule* is a pair (t, u) of terms, possibly allowing variables. It differs from an equation in that it contains a left-to-write orientation.

We return to the set E of equations in Figure 2. We construct a rewriting system \mathcal{R} by taking the pairs (ℓ, r) whenever $\ell = r$ is an equation in the system. That is, we orient all equations in Figure 2 left-to-right, as they are written. In particular, we orient the associative law for $+$ thus: $(x + y) + z \rightarrow x + (y + z)$.

Proposition 3.1 *This term-rewriting system \mathcal{R} is terminating and confluent.*

To say that \mathcal{R} is terminating means there are no infinite sequences of applications of the rules. To say that it is confluent: If t rewrites* to both u and v , then u and v can further rewrite* to a common term w .

The termination and confluence results for term rewriting system in this paper are discussed briefly in Appendix A.

It follows from Proposition 3.1 and Theorem 2.1 that every term t has a unique *normal form* $\text{nf}(t)$. By Theorem 2.1, the word problem for ground terms in this signature is decidable, the initial algebra is the set of normal forms, and this algebra is computable.

Definition 3.2 *A term t is a quine if it is a normal form and $t @ \mathbf{e} \equiv t$. Terms t and u are twins if they are distinct normal forms such that $t @ \mathbf{e} \equiv u$ and $u @ \mathbf{e} \equiv t$.*

Example 3.3 [Standard presentation of quines] The main example is $\mathbf{d} @ \mathbf{d}$. Note that

$$(\mathbf{d} @ \mathbf{d}) @ \mathbf{e} \rightarrow \mathbf{d} @ (\mathbf{e} + \mathbf{d}) \rightarrow \mathbf{d} @ \mathbf{d}.$$

In addition, \mathbf{e} is trivially a quine.

Example 3.4 The term $t = \mathbf{d} @ (\mathbf{d} + \mathbf{d})$ is a normal form. But $t @ \mathbf{e}$ is not a normal form, and indeed we have

$$t @ \mathbf{e} \rightarrow (\mathbf{d} + \mathbf{d}) @ (\mathbf{e} + (\mathbf{d} + \mathbf{d})) \rightarrow \mathbf{d} @ (\mathbf{d} @ (\mathbf{d} + \mathbf{d})).$$

This last term is a normal form.

The term $u = (\mathbf{d} + \mathbf{d})$ is a normal form. But $u @ \mathbf{e}$ is not a normal form, and instead $u @ \mathbf{e} \rightarrow \mathbf{d} @ (\mathbf{d} @ \mathbf{e}) \rightarrow \mathbf{d} @ \mathbf{e} \rightarrow \mathbf{e}$.

The main results in this section are the characterization of normal forms, Proposition 3.7, and the resulting results on quines and twins in this language, Theorems 3.14 and 3.15.

Proposition 3.5 *If the term t is in normal form, then every subterm of t of the form $u @ v$ has $u = \mathbf{d}$. In particular, if t is an $@$ -term which is a normal form, then there is a normal form u such that $t = \mathbf{d} @ u$.*

Proof. By induction on the term t . We can assume that t is a term in normal form, and that t is an $@$ -term $u @ v$. If $u = \mathbf{e}$, then t would not be a normal form. The same holds if u were a $+$ -term. So u must either be \mathbf{d} (and we are done), or u is an $@$ -term. Since t is in normal form, so is u . By induction hypothesis, u is of the form $\mathbf{d} @ \mathbf{w}$. But then t is $(\mathbf{d} @ \mathbf{w}) @ v$, and this is a redex. So we have a contradiction in this case. \square

Proposition 3.5 provides a necessary condition for a term to be a normal form: in every application subterm $u @ v$, the term u must be \mathbf{d} . This condition is sufficient, as the next result shows. We first introduce some notation for later use.

Definition 3.6 Let N , N_+ , $N_{\textcircled{a}}$ be the smallest sets of terms such that

- (i) $\mathbf{d}, \mathbf{e} \in N$.
- (ii) $N_+ \cup N_{\textcircled{a}} \subseteq N$.
- (iii) If $t \in N \setminus \{\mathbf{e}\}$, then $\mathbf{d} @ t$ belongs to $N_{\textcircled{a}}$.
- (iv) If $n \geq 2$ and $t_1, \dots, t_n \in N_{\textcircled{a}} \cup \{\mathbf{d}\}$, then $t_1 + (t_2 + \dots + (t_{n-1} + t_n) \dots) \in N_+$.

Observe that N is the disjoint union of the three sets $\{\mathbf{e}, \mathbf{d}\}$, $N_{\textcircled{a}}$, and N_+ .

Proposition 3.7 N is the set of normal forms.

Proof. It is easy to check that all terms in N are normal forms. In the other direction, we check by induction on the term t that (a) if t is a normal form and an \textcircled{a} -term, then $t \in N_{\textcircled{a}}$, and (b) if t is a normal form and a $+$ -term, then $t \in N_+$. The first assertion is stated in Proposition 3.5. For the second, suppose that we have a normal form $t = u + v$. Neither u nor v can be \mathbf{e} , since t is a normal form. And u cannot be a $+$ -term, due to our orientation of associativity. So u is either \mathbf{d} or an \textcircled{a} -term. As for v , if $v = \mathbf{d}$, then $t = u + v \in N_+$. If v is an \textcircled{a} -term, then $v \in N_{\textcircled{a}}$ by induction hypothesis, and we are done. If v is a $+$ -term, then by induction hypothesis it is $w_1 + (w_2 + \dots + (w_{m-1} + w_n) \dots)$ where each $w_i \in N_{\textcircled{a}} \cup \{\mathbf{d}\}$. Since $t = u + (w_1 + (w_2 + \dots + (w_{m-1} + w_n) \dots))$, $t \in N_+$. \square

Proposition 3.8 If t is a sum of terms u_1, u_2, \dots, u_k in any parenthesization, and $t \equiv \mathbf{e}$, then for all i , $u_i \equiv \mathbf{e}$.

Proof. We show by induction on n that if any finite sum of terms $u_1 + u_2 + \dots + u_k$ (in any parenthesization) rewrites to \mathbf{e} in n steps, then $u_i \equiv \mathbf{e}$ for $1 \leq i \leq k$. For $n = 0$, the result is immediate. Assume our result for n , and suppose that $u_1 + \dots + u_k$ rewrites to \mathbf{e} in $n + 1$ steps. If the first step of rewriting involves subterms of some u_j , obtaining u'_j , consider the resulting term $u_1 + \dots + u'_j + \dots + u_k$. By induction hypothesis, $u_i \equiv \mathbf{e}$ for all $i \neq j$, and also $u'_j \equiv \mathbf{e}$. But since $u_j \equiv u'_j$, we also have $u_j \equiv \mathbf{e}$. The other option in the induction step is when we use one of the first three rules in Figure 2: $x + \mathbf{e} = x$, $\mathbf{e} + x = x$, or associativity of $+$. In all cases, the result follows easily from induction hypothesis. \square

Proposition 3.9 For all normal forms $t \neq \mathbf{e}$ and all normal forms u :

- (i) \mathbf{e} does not occur in t .
- (ii) $\text{nf}(t @ u) \in \{\mathbf{e}\} \cup N_{\textcircled{a}}$.
- (iii) If $\text{nf}(t @ u) = \mathbf{e}$, then $u = \mathbf{e}$.

Proof. Part (i) follows immediately from Proposition 3.7. We show the other two parts (together) by induction on the term t .

For $t = \mathbf{e}$, there is nothing to do. For $t = \mathbf{d}$, $t + u$ cannot rewrite to \mathbf{e} . In the other parts, recall that if u is a normal form and $u \neq \mathbf{e}$, then $\mathbf{d} @ u \in N_{\textcircled{a}}$.

Suppose that t is an \textcircled{a} -term, say $t = \mathbf{d} @ v$. Assume that t is a normal form and hence is not \mathbf{e} . Assume parts (ii) and (iii) for all subterms of t , and fix u . Then $t @ u = (\mathbf{d} @ v) @ u \equiv v @ (u + v)$. Our induction hypothesis applies to v , since it is a subterm of t . So

$$\text{nf}(t @ u) = \text{nf}(v @ (u + v)) \in \{\mathbf{e}\} \cup N_{\textcircled{a}}.$$

And if $\text{nf}(t @ u) = \mathbf{e}$, then by induction hypothesis on v , $u + v \equiv \mathbf{e}$. By Proposition 3.8, $u = \mathbf{e}$. This is what we want.

Finally, suppose that t is a $+$ -term normal form, say $t = v_1 + (v_2 + v_3)$. (For $n \geq 3$ in the definition of N_+ , the argument is similar.) Fix u . Assume that $\text{nf}(t @ u) = \mathbf{e}$. Then

$$t @ u = (v_1 + (v_2 + v_3)) @ u \equiv v_3 @ (v_2 @ (v_1 @ u)).$$

Our induction hypothesis applies to v_1 , v_2 , and v_3 . We have $\text{nf}(v_3 @ (v_2 @ (v_1 @ u))) = \mathbf{e} \in \{\mathbf{e}\} \cup N_{\textcircled{a}}$. By

applying our induction hypothesis to v_3 , we see that $v_2 @ (v_1 @ u) = \mathbf{e}$. Then we apply the induction hypothesis to v_2 , and then v_1 . We see that $u = \mathbf{e}$, as desired. \square

3.4 Characterization of quines

Definition 3.10 *The \mathbf{d} -count of a term $\mathbf{d}\text{-ct}(t)$ is the number of \mathbf{d} 's in it. It is given by the following recursion:*

$$\begin{aligned} \mathbf{d}\text{-ct}(\mathbf{e}) &= 0 & \mathbf{d}\text{-ct}(t + u) &= \mathbf{d}\text{-ct}(t) + \mathbf{d}\text{-ct}(u) \\ \mathbf{d}\text{-ct}(\mathbf{d}) &= 1 & \mathbf{d}\text{-ct}(t @ u) &= \mathbf{d}\text{-ct}(t) + \mathbf{d}\text{-ct}(u) \end{aligned}$$

Proposition 3.11 *The only normal form t with $\mathbf{d}\text{-ct}(t) = 0$ is \mathbf{e} , and the only normal form t with $\mathbf{d}\text{-ct}(t) = 1$ is \mathbf{d} .*

Lemma 3.12 *Let t be a term in which \mathbf{e} does not occur (not necessarily a normal form). Then $\mathbf{d}\text{-ct}(\mathbf{nf}(t)) \geq \mathbf{d}\text{-ct}(t)$.*

Proof. Fix t with no occurrences of \mathbf{e} . We go from t to its normal form by applying the rewrite rules. Each application of a rule maintains the $\mathbf{d}\text{-ct}$, with two exceptions: $\mathbf{d} @ \mathbf{e} \rightarrow \mathbf{e}$, and $(\mathbf{d} @ x) @ y \rightarrow x @ (y + x)$. The first of these rules cannot apply since we start with a term having no occurrences of \mathbf{e} . As for the rule $(\mathbf{d} @ x) @ y \rightarrow x @ (y + x)$, we lose the first \mathbf{d} on the left, but this is made up by the extra x on the right. And x , like all subterms of t , must have at least one \mathbf{d} . So in going from t to $\mathbf{nf}(t)$, we perform rewrites which maintain or increase the \mathbf{d} -count. \square

Remark 3.13 The hypothesis that \mathbf{e} does not occur in t is needed. For example $\mathbf{d} @ \mathbf{e}$ has a \mathbf{d} -count of 1, yet its normal form is \mathbf{e} .

Theorem 3.14 [*uniqueness of non-empty quines*] *If t is a quine, then either $t = \mathbf{e}$ or else $t = \mathbf{d} @ \mathbf{d}$.*

Proof. Let t be a normal form such that $\mathbf{nf}(t @ \mathbf{e}) = t$. Let us also assume that t is not \mathbf{e} . Then t cannot be \mathbf{d} , since $\mathbf{d} @ \mathbf{e} = \mathbf{e}$. And t cannot be a $+$ -term by Proposition 3.9. So t is an $@$ -term, say $\mathbf{d} @ u$ with u a normal form and $u \neq \mathbf{e}$ (see Proposition 3.5). Now

$$\mathbf{d} @ u = t = \mathbf{nf}(t @ \mathbf{e}) = \mathbf{nf}((\mathbf{d} @ u) @ \mathbf{e}) = \mathbf{nf}(u @ u).$$

Notice that $u @ u$ need not be a normal form. But since $u \neq \mathbf{e}$ is a normal form, \mathbf{e} does not occur in u by Proposition 3.9(i). Hence \mathbf{e} does not occur in $u @ u$. Applying the \mathbf{d} -count function and using Lemma 3.12, we see that

$$1 + \mathbf{d}\text{-ct}(u) = \mathbf{d}\text{-ct}(\mathbf{d} @ u) = \mathbf{d}\text{-ct}(\mathbf{nf}(u @ u)) \geq \mathbf{d}\text{-ct}(u @ u) = 2 \cdot \mathbf{d}\text{-ct}(u).$$

Therefore $\mathbf{d}\text{-ct}(u) \leq 1$. It follows that u is either \mathbf{d} or \mathbf{e} , by Proposition 3.11. As we know, $u \neq \mathbf{e}$. Thus $u = \mathbf{d}$, and so $t = \mathbf{d} @ u = \mathbf{d} @ \mathbf{d}$, as desired. \square

Proposition 3.15 *In the language of this section, there are no cycles of length ≥ 2 .*

Proof. Let $n \geq 2$. In this proof, the letter i ranges over $\{0, 1, \dots, n-1\}$, and $i+1$ taken mod n . Assume that t_0, \dots, t_{n-1} are distinct normal forms, and $t_i @ \mathbf{e} \equiv t_{i+1}$. It is clear that no t_i is \mathbf{e} . By Proposition 3.9(ii), no t_i is a $+$ -term. Thus, each t_i is an $@$ -normal form. So we have terms u_0, \dots, u_{n-1} in normal form such that $t_i = \mathbf{d} @ u_i$. Since $t_i \neq \mathbf{e}$, we also have $u_i \neq \mathbf{e}$. For all i ,

$$\mathbf{d} @ u_{i+1} = t_{i+1} \equiv t_i @ \mathbf{e} = (\mathbf{d} @ u_i) @ \mathbf{e} \equiv u_i @ u_i$$

Thus $\mathbf{d} @ u_{i+1} = \mathbf{nf}(u_i @ u_i)$. So

$$1 + \mathbf{d}\text{-ct}(u_{i+1}) = \mathbf{d}\text{-ct}(\mathbf{d} @ u_{i+1}) = \mathbf{d}\text{-ct}(\mathbf{nf}(u_i @ u_i)) \geq \mathbf{d}\text{-ct}(u_i @ u_i) = 2\mathbf{d}\text{-ct}(u_i)$$

Taking the sum of the last line for $i = 0, \dots, n - 1$ gives

$$n + (\mathbf{d}\text{-ct}(u_0) + \dots + \mathbf{d}\text{-ct}(u_{n-1})) \geq 2(\mathbf{d}\text{-ct}(u_0) + \dots + \mathbf{d}\text{-ct}(u_{n-1}))$$

Hence $\mathbf{d}\text{-ct}(u_0) + \dots + \mathbf{d}\text{-ct}(u_{n-1}) \leq n$. Since each \mathbf{d} -count is a natural number, and none can be 0, they are all 1. But then all u_i are \mathbf{d} . It follows that for all i , $t_{i+1} = \mathbf{d}@\mathbf{d}$. This contradicts the assumption that the terms t_0, \dots, t_{n-1} are distinct normal forms and $n \geq 2$. \square

3.5 Further details on a computable model of the equations which treats $@$ as Kleene application

In this section, we continue the discussion of a model of our set E equations that we started in Section 3.1. The goal is to have a model where $t@u$ is interpreted as $\varphi_t(u)$ and where $+$ is concatenation of words or something close. (The reason that we say “something close” here is that the interpretation of $+$ in our model below is not literally concatenation, due to the treatment of the comma.) This stands in contrast with the term model of E , where the interpretation of the application function on two terms t and u is simply the term $t@u$.

We are discussing the “programming language” P presented as pseudocode. This is mainly for expository purposes. The results here apply to the mathematically precise language $\mathbf{1}\#$ presented by the author in [10] and also implemented in courseware on computability theory. The main things that one would need to know about $\mathbf{1}\#$ are that (i) it is Turing-complete; (ii) there are explicit programs corresponding to the constants mentioned in this paper; (iii) programs are strings, and the concatenation of any two programs is a program (so that $+$ is associative); (iv) the equational laws under study hold.

We write T for the set of terms built from \mathbf{d} , \mathbf{e} , $+$ and $@$. We are only interested in the ground terms, so there are no variables. We have a partial function $*$: $T \rightarrow P$ given by

$$\begin{aligned} \mathbf{d}^* &= \text{diag} & (t + u)^* &= t^* + u^* \\ \mathbf{e}^* &= \varepsilon & (t @ u)^* &= \llbracket t^* \rrbracket(u^*) \end{aligned}$$

Here ε is the empty word, and on the right in the third equation, we again use $+$ for concatenation in P . The reason we emphasize that this is a partial function is that on the right we have application of a program to an argument, and in general this is partial. The main point is that nevertheless, this function $*$ is total. That is application of a program to an argument is in general partial, but for programs in the image of the interpretation map $*$, we need not worry. Let

$$\begin{aligned} A_0 &= \{x : x \text{ is a program of } P\} \\ A_{n+1} &= \{x \in A_n : \text{for all } y \in A_n, \llbracket x \rrbracket(y) \text{ is defined and belongs to } A_n\} \\ A_\omega &= \bigcap_n A_n \end{aligned}$$

Note that A_1 is the set of programs which compute total functions (and indeed output programs), A_2 is the set of programs which take elements of A_1 to elements of A_1 , etc. Observe that $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$. In fact, the inclusions are all strict.

Proposition 3.16 *Each set A_n is closed under $+$, where $+$ is concatenation of programs.*

Proof. A_0 is closed under $+$ because the concatenation of two programs is always a program. For $n + 1$, let $x, y \in A_{n+1}$, and let $z \in A_n$. Then $\llbracket x + y \rrbracket(z) \simeq \llbracket y \rrbracket(\llbracket x \rrbracket(z))$. Since $x \in A_{n+1}$, $\llbracket x \rrbracket(z)$ is defined and belongs to A_n . And then since $y \in A_{n+1}$, $\llbracket y \rrbracket(\llbracket x \rrbracket(z))$ is defined and belongs to A_n . Thus $\llbracket x + y \rrbracket(z)$ has this same property. \square

Lemma 3.17 *For all $t \in T$, t^* is defined and is an element of A_ω . In particular, $*$ is a total function.*

Proof. By induction on T . First, $\mathbf{e}^* = \varepsilon$ belongs to A_ω , since for $x \in A_n$, $\llbracket \varepsilon \rrbracket(x) = x \in A_n$.

Let us show next that $\mathbf{d}^* = \mathit{diag}$ belongs to A_n for all n . We use induction on n . This is clear for $n = 0$ since diag is a program. Assume that $\mathit{diag} \in A_n$. To check that $\mathit{diag} \in A_{n+1}$, let $x \in A_n$; we show that $\llbracket \mathit{diag} \rrbracket(x) \in A_n$. If $n = 0$, this again is clear because $\llbracket \mathit{diag} \rrbracket$ is a total function on programs. So we assume that $n > 0$. Let $y \in A_{n-1}$. Now

$$\llbracket \llbracket \mathit{diag} \rrbracket(x) \rrbracket(y) = \llbracket x \rrbracket(y + x)$$

Since x and y belong to A_{n-1} , the program $y + x$ also belongs to A_{n-1} by Proposition 3.16. Since $x \in A_n$, $\llbracket x \rrbracket(y + x) \in A_{n-1}$. Therefore, $\llbracket \llbracket \mathit{diag} \rrbracket(x) \rrbracket(y) \in A_{n-1}$. Since y is arbitrary in A_{n-1} , $\llbracket \mathit{diag} \rrbracket(x) \in A_n$. And since x is arbitrary in A_n , $\mathit{diag} \in A_{n+1}$. This completes the induction and shows that $\mathit{diag} \in A_\omega$.

Next, assume about t and u that t^* and u^* belong to A_ω . Using Proposition 3.16, we see that $(t + u)^* = t^* + u^*$ belongs to A_ω .

Finally, assume about t and u that t^* and u^* belong to A_ω . We show that $(t @ u)^*$ also belongs to A_ω . Fix n . Then $u^* \in A_n$ and $t^* \in A_{n+1}$. So $t^*(u^*) \in A_n$. This for all n shows what we want: $t^*(u^*) = (t @ u)^* \in A_\omega$. \square

Example 3.18 [Yang [17]] Let terms p_n be defined as follows:

$$\begin{aligned} p_0 &= \mathbf{d} @ (\mathbf{d} + \mathbf{d}) \\ p_{n+1} &= p_n @ \mathbf{e} \end{aligned}$$

Yang observed that this gives a sequence of terms $p_0 \rightarrow p_1 \rightarrow \dots$ which are different, where the arrow is from (2). Here is a bit more on this. First, one shows that $\mathit{nf}(p_n) \neq \mathbf{d} @ \mathbf{d}$ for all n . Since \rightarrow has no cycles, they must all be different. But we are not interested in these as *terms* but rather as actual programs. That is, we fix an interpretation $^* \rightarrow \mathcal{L}$ for some programming language \mathcal{L} . Each p_n^* is a total function, as we know. In particular, p_n^* is well-defined. To know that they are all different, we would like to know that the interpretation function $^* \rightarrow \mathcal{L}$ is one-to-one. But this is not something which we know at this time. Instead, special purpose counting arguments would be needed to show that the interpretations of the terms p_n are all different programs.

3.6 A (non-computable) model based on the natural numbers where again application is interpreted as Kleene application

We exhibit a different model, one based on the natural numbers but where we identify numbers m and n if $\llbracket m \rrbracket = \llbracket n \rrbracket$ as partial functions. This makes our model non-computable.

Fix numbers i, b, b', s_1^1 and d such that

$$\begin{aligned} \llbracket i \rrbracket(x) &= x \\ \llbracket \llbracket b \rrbracket(x, y) \rrbracket(z) &= \llbracket x \rrbracket(\llbracket y \rrbracket(z)) \\ \llbracket \llbracket b' \rrbracket(x, y) \rrbracket(z) &= \llbracket y \rrbracket(\llbracket x \rrbracket(z)) \\ \llbracket \llbracket s_1^1 \rrbracket(x, y) \rrbracket(z) &= \llbracket x \rrbracket(y, z) \\ \llbracket d \rrbracket(x) &= \llbracket b' \rrbracket(\llbracket s_1^1 \rrbracket(b, x), x) \end{aligned}$$

Then interpret \mathbf{e} by i , \mathbf{d} by d , $x + y$ by $\llbracket b' \rrbracket(x, y)$, and $@$ by $n @ m = \llbracket n \rrbracket(m)$.

The main reason that we want the identification that we mentioned above is that certain equations of interest do not hold on the nose but only up to this identification. For example, $\llbracket b \rrbracket(x, y) \equiv \llbracket b' \rrbracket(y, x)$, but the two sides are not literally equal.

It is straightforward to verify the equations of interest. The main verification: $(\mathbf{d} @ x) @ y = x @ (y + x)$.

$x + e = x$	$d @ x = (w @ x) + x$	$w @ e = e$
$e + x = x$	$e @ x = x$	
$(x + y) + z = x + (y + z)$	$w @ (x + y) = (w @ x) + (w @ y)$	
$(x + y) @ z = y @ (x @ z)$	$w @ x @ y = y + x$	

Fig. 3. The set E of equations used for the language with constants d , e , and w , and operations $+$ and $@$.

$$\begin{aligned}
(d @ x) @ y &= \llbracket [d](x) \rrbracket (y) \\
&= \llbracket [b'](\llbracket [s_1^1](b, x), x \rrbracket)(y) \rrbracket \\
&= \llbracket [x](\llbracket [s_1^1](b, x)(y) \rrbracket) \rrbracket \\
&= \llbracket [x](\llbracket [b](x, y) \rrbracket) \rrbracket \\
&\equiv \llbracket [x](\llbracket [b'](y, x) \rrbracket) \rrbracket \\
&= x @ (y + x)
\end{aligned}$$

4 Adding a constant w for the “write” operation

We could at this point reconsider the constant symbol d and replace it with the constant w along with the relevant equations. If we did this, the language would not be expressive enough for the kinds of questions which we ask in this paper. Specifically, there would not be any quine programs (see Corollary 4.8).

We therefore *add* w on top of the language which we saw in the previous section. We adopt the equations in Figure 3.

Remark 4.1 Previously we had the equation $(d @ x) @ y = x @ (y + x)$. This equation is not found in Figure 3. However, it is derivable:

$$(d @ x) @ y = ((w @ x) + x) @ y = x @ ((w @ x) @ y) = x @ (y + x).$$

Example 4.2 [Slattery’s twins] In contrast to what we saw in Proposition 3.15 for the language with d but not w , in the language that also has w , there are twins. Let $p = w @ (d @ (d + w))$ and $q = d @ (d + w)$. Then

$$\begin{aligned}
p @ e &= (w @ (d @ (d + w))) @ e \equiv d @ (d + w) = q. \\
q @ e &= (d @ (d + w)) @ e \equiv (d + w) @ (d + w) \equiv w @ (d @ (d + w)) = p.
\end{aligned}$$

The normal forms $t = \text{nf}(p)$ and $u = \text{nf}(q)$ are shown below. They are distinct, so we have twins:

$$\begin{aligned}
t &= w @ (w @ d) + w @ (w @ w) + w @ d + w @ w. \\
u &= w @ d + w @ w + d + w.
\end{aligned}$$

This pair of twins is due to Austin Slattery [13].

Example 4.3 Another pair of twins is

$$\begin{aligned}
t &= w @ w + w @ (w @ d) + w + w @ d \\
u &= w @ w + w @ (w @ d) + d
\end{aligned}$$

Let us translate these to our stylized form of English commands P from Section 3.1. First, t^* is

write “write the instructions to write what is in the register”,
write “write “write the instructions to write what is in the register in front of it” ”,
write the instructions to write what is in the register,
write “write the instructions to write what is in the register in front of it”

Second, u^* is

write “write the instructions to write what is in the register”,
write “write “write the instructions to write what is in the register in front of it” ”,
write the instructions to write what is in the register in front of it

Example 4.4 [Slattery’s multiples] Slattery also observed that his twins generalize to cycles of all lengths. Here is the construction of a 3-cycle. Let

$$p = (d + w + w) @ (d + w + w),$$

and observe that $p \equiv w @ (w @ (d @ (d + w + w)))$. Then

$$\begin{aligned} p @ e & \equiv w @ (d @ (d + w + w)) \\ (w @ (d @ (d + w + w))) @ e & \equiv d @ (d + w + w) \\ (d @ (d + w + w)) @ e & \equiv p \end{aligned}$$

Definition 4.5 *As we did with our previous set of equations, we construct a rewriting system using the relation $l \rightarrow r$ whenever $l = r$ is an equation in Figure 3. We overload our notation to call this rewriting system \mathcal{R} .*

Proposition 4.6 *This term-rewriting system \mathcal{R} is terminating and confluent.*

In a normal form, each application term $t @ u$ has $t = w$. Thus, every normal form t is a finite sum of terms, each of the following three forms: d , w , or $w @ t'$, where t' is again a normal form other than e .

Theorem 4.7 *Let t and u be normal forms such that $t @ e \equiv u$ and $u @ e \equiv t$. Then one of the following holds:*

- (i) $t = e = u$.
- (ii) $t = (w @ d) + d = u$. In other words, $t \equiv d @ d \equiv u$.
- (iii) $t = w @ w + w @ (w @ d) + w + w @ d$, and $u = w @ w + w @ (w @ d) + d$, or vice-versa.
- (iv) $t = w @ (w @ d) + w @ (w @ w) + (w @ d) + (w @ w)$ and $u = (w @ d) + (w @ w) + d + w$, or vice-versa. In other words, t and u are Slattery’s twins.

The proof is a long case-by-case analysis, and we lack the space for it here.

Corollary 4.8 *In the language with only w , e , $+$ and $@$, there are no non-trivial quine programs.*

5 A next step: adding constants u for the universal function and s for s_1^1

Figure 2 formulates a larger language than what we studied in the main body of the paper. It includes a constant for the s_1^1 function (expressed in curried form, as one would expect here) and two constants for different presentations of the universal function in computability theory. The most natural way to formulate this is via what we wrote as u , with the equation $u @ x @ y = x @ y$. And for this paper, we also want a constant u_0 representing running a program on “empty registers”. The equation here is $u_0 @ x = x @ e$.

What we know about the associated rewriting system is that it is confluent, and without the equation for u_0 , it is also terminating. This means that the word problem is decidable for it. But adding u_0 and its equation results in a term rewriting system which is (surprisingly) non-terminating:

$$\begin{aligned} (\mathbf{d} @ (\mathbf{d} + \mathbf{u})) @ \mathbf{e} &\rightarrow (\mathbf{d} + \mathbf{u}) @ (\mathbf{e} + (\mathbf{d} + \mathbf{u})) \rightarrow \mathbf{u} @ (\mathbf{d} @ (\mathbf{e} + (\mathbf{d} + \mathbf{u}))) \\ &\rightarrow \mathbf{u} @ (\mathbf{d} @ (\mathbf{d} + \mathbf{u})) \rightarrow (\mathbf{d} @ (\mathbf{d} + \mathbf{u})) @ \mathbf{e} \end{aligned}$$

We do not know if the word problem remains decidable.

6 Conclusion and Future Work

We proposed several sets of equations related to “self-expression” by programs, or more generally to the expression of one program by another. The overall goal was to have an equational axiomatization of very simple but very useful parts of computability theory, and then to study the systems of equations which emerge. The main results concerned the equations in Figures 2 and 3. We observed that in both of these, the word problem is decidable, calling on results from term rewriting theory. Indeed, the natural rewriting systems coming from these sets of equations are terminating and confluent. Then we studied the normal forms in both systems, obtaining a few results on self-writing programs and cycles.

The decidability results in this paper contrast with those in stronger systems like combinatory logic. From our point of view, this is because CL usually starts with the \mathbf{s} and \mathbf{k} combinators, and thus it starts with everything one would need to represent every computable partial function. What we did in this paper goes the other way, starting only with what is necessary to study the phenomena of interest. What we lose in expressive power is (we hope) recovered in applicability and interest of the results. Be that as it may, what we are doing fits a pattern in logic and theoretical computer science of studying “light” systems, where again one aims for logical systems which are both useable and tractable.

We were able to classify the quines and cycles in our equational logics. The uniqueness of quine programs is at first surprising. Some have felt that it must be wrong, since in general there are no uniqueness results for anything in computability theory. All of the constructions can be modified in small ways. We are not saying that quine programs are unique, of course. We are saying that in our equational logics of either Figures 2 or 3, there is just one term q such that $q @ \mathbf{e} \equiv q$. Moving to a real programming language \mathcal{L} , or to indices of computable functions, means defining an interpretation $*$: $T \rightarrow \mathcal{L}$, where T is the set of terms under discussion. It is this interpretation function which is so non-unique. There are many ways to modify one such function. In a sense, what we are saying about quine programs is that if one wants to write one using a *fixed* diagonal function and allows composition of programs, and nothing else, then there is only one way to do it. Even if one adds the “write” construct, there is just one quine program, up to equivalence in a simple equational logic. Moreover, adding that “write” construct gives cycles of all lengths; these would not exist without “write.” All of these results were based on term rewriting theory, and we know of no other way to obtain them.

Our results suggest a number of open questions.

Returning to our discussion of quine programs in everyday programming languages, we wonder whether a language which does *not* allow programs to apply to other programs can possibly have an interpretation of the diagonal operator *diag* in our sense. But such a language would still have quine programs (and more). So one wonders whether there is a different form of equational reasoning that could illuminate the self-referential aspects of such languages. Admittedly, this question is vague.

We formulated a set of equations in Figure 2 and we know about the termination and confluence properties of the associated term rewriting system. (It is confluent, but the one equation $u_0 @ x = x @ \mathbf{e}$ messes up the termination. Without this, the system is terminating.) But we did not go further with this set of equations, and it is open to see whether characterization results like Theorems 3.14 and 3.15 hold for this bigger logic.

Another open question concerns the computable model which we discussed in Section 3.5. We showed that one can interpret terms in our first equational logic in a pseudocode language, and that the inter-

pretation of every term is a total computable function (and more). We conjecture that this same results holds for arbitrary programming languages; as soon as one has a Turing-complete language, we believe that it should interpret our logical systems. Moreover, the interpretation function which we studied should be one-to-one. One precise formulation would be that the partial combinatory algebra built from Kleene application should have the initial algebra of our equations as a subalgebra, and Kleene application in this subalgebra should be total. We leave this for future work.

A final open question concerns extensional models of our equations. In CL, extensionality is the principle that if x is a variable, from $tx = ux$, one infers $t = u$. This can be formulated for our systems. We do not know if the decidability results here extend to the system that also includes extensionality.

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Appendix A: Details on the termination and confluence

It is well-known that the rewriting system coming from the **s** and **k** combinators is confluent (Church-Rosser) but not terminating. These central results do not help with fragments such as the ones in this paper: we know of no general results that allow one to infer the confluence of one rewrite system from the confluence of a larger system, and of course we are after positive results on termination.

All of the termination/confluence results in this paper were shown by running programs. We used AProVE2023 [4] in connection with the equational system in Figure 2. AProVE2023 tells us that termination can be shown with a recursive path ordering (RPO) using the quasi-precedence $@ > [+ , \mathbf{e}, \mathbf{d}]$. Other programs showed the termination using other general results. Adding **u** with the rule $(\mathbf{u} @ x) @ y \rightarrow x @ y$ does not destroy the termination; one uses the RPO with $@ > \mathbf{e} > [+ , \mathbf{d}]$, $\mathbf{u} > [+ , \mathbf{d}]$. Adding $\mathbf{u} @ x \rightarrow x @ \mathbf{e}$ results in a loss of termination, as we saw in Section 5; the example there was also found by AProVe. The confluence results are also obtained by machine. Here the most informative results come from ConCon [15] and from TRS.tool [7]. None of our systems have critical pairs, and they are locally confluent.