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A Complete $\mathcal{V}$-Equational System for Graded λ-Calculus

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Abstract

Modern programming frequently requires generalised notions of program equivalence based on a metric or a similar structure. Previous work addressed this challenge by introducing the notion of a $\mathcal{V}$-equation, *i.e.* an equation labelled by an element of a quantale $\mathcal{V}$, which covers *inter alia* (ultra-)metric, classical, and fuzzy (in)equations. It also introduced a $\mathcal{V}$-equational system for the *linear* variant of λ-calculus where any given resource must be used exactly once.
In this paper we drop the (often too strict) linearity constraint by adding graded modal types which allow multiple uses of a resource in a controlled manner. We show that such a control, whilst providing more expressivity to the programmer, also interacts more richly with $\mathcal{V}$-equations than the linear or Cartesian cases. Our main result is the introduction of a sound and complete $\mathcal{V}$-equational system for a λ-calculus with graded modal types interpreted by what we call a *Lipschitz exponential comonad*. We also show how to build such comonads canonically via a universal construction, and use our results to derive graded metric equational systems (and corresponding models) for programs with timed and probabilistic behaviour.

Keywords: λ-calculus, graded modal type, quantitative equational theory, enriched category theory.

1 Introduction

This paper tackles the challenge of reasoning about program equivalence in computational paradigms with an intrinsic quantitative nature, such as timed and probabilistic computation. This usually calls for notions of program equivalence based on a quantity (often a metric), *in lieu* of the sharp, binary ones relating classical programs. For example, instead of checking whether two programs terminate *exactly* at the same time one might be more interested in checking whether they terminate with a small difference between their execution times. Similarly, on the probabilistic side, it makes sense to consider that two Bayesian inference algorithms are equivalent if they agree up to some small (total variation) error ε when sampling from the same target posterior distribution. In order to reason in this way, [13] introduced the notion of a $\mathcal{V}$-equation, *i.e.* an equation labelled by an element of a quantale $\mathcal{V}$, that serves as an abstract notion of 'quantitative equality'. This covers, for example, (ultra-)metric and fuzzy (in)equations, among others. Additionally [13] presented a $\mathcal{V}$-equational system for the *linear* version of λ-calculus which imposes that any given resource must be used exactly once.

The aim of this work is to overcome this linearity constraint whilst retaining the ability to reason quantitatively about program equivalence. We do so by adding *graded modal types* [23,19,44] (a way of

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permitting multiple uses of a given resource) to the aforementioned $\mathcal{V}$-equational framework of linear λ-calculus [13]. The result is a compromise between standard, non-linear λ-calculus which is to some degree incompatible with quantitative reasoning (see the negative results of [33, §6]) and linear λ-calculus which can be combined with quantitative reasoning [13] but is cumbersome for many non-linear applications.

Let us illustrate this compromise with a simple example that involves metric equations [40] and timed computation [13]. Consider a ground type X and a signature $\{\mathtt{wait_n} : X \to X \mid n \in \mathbb{N}\}$ of wait calls – intuitively, a term $\mathtt{wait_n}(x)$ reads as "add a latency of n seconds to computation x". As discussed in [13], a series of metric equations arise naturally from this computational paradigm. For example,

$$\lambda x.\, \mathtt{wait}_1(x) =_1 \lambda x.\, \mathtt{wait}_2(x) \tag{1}$$

states that when fed the same argument these λ-terms yield computations whose execution times differ by at most one second. Now, as a useful principle that underpins compositionality we would like that for all λ-terms u the application function $v \mapsto u\, v$ satisfies the implication $v =_q w \Rightarrow u\, v =_q u\, w$, *i.e.* it is non-expansive w.r.t. distances between programs. This is impossible in the Cartesian setting, because u may contain multiple ocurrences of a variable (corresponding to multiple uses of a given resource). Let u for example be $\lambda f.\, \lambda y.\, f\,(f\,y)$. Then $u\,(\lambda x.\, \mathtt{wait}_1(x))$ corresponds to an execution time of two seconds and $u\,(\lambda x.\, \mathtt{wait}_2(x))$ to four seconds, a two-second difference that violates the implication for (1). The graded setting explored in this paper serves as middleground between the linear and Cartesian cases: it increases distances proportionally to the number of times a resource is usable and at the same time forbids u from using a resource more times than stipulated. Specifically for the case just presented one can mark $\lambda x.\, \mathtt{wait}_1(x)$ (resp. $\lambda x.\, \mathtt{wait}_2(x)$) to be usable *precisely twice*, via a 'promotion construct' $!_2(-)$, and according to our graded equational system deduce the metric equation,

$$!_2\big(\lambda x.\, \mathtt{wait}_1(x)\big) =_{1+1} !_2\big(\lambda x.\, \mathtt{wait}_2(x)\big)$$

We then use the graded typing system to ensure u uses the received argument precisely twice. We will see that this ensures the non-expansiveness of the application function – actually of the more general case $(u, v) \mapsto u\, v$ – amongst other benefits.

Contributions and outline. We present a sound and complete $\mathcal{V}$-equational system for a graded λ-calculus. The corresponding interpretation is based on symmetric monoidal closed categories enriched over 'generalised metric spaces' and equipped with a *Lipschitz exponential comonad*, a natural extension of the concept of graded exponential comonad [19,30] to the setting of $\mathcal{V}$-equations. Furthermore, we show how to canonically build Lipschitz exponential comonads over symmetric monoidal closed categories that satisfy mild conditions. The construction is inspired by [41], and based on the notion of a cofree graded commutative comonoid together with a certain kind of enriched limit.

§2 introduces a graded λ-calculus and an equational system that characterises term equivalence. This calculus fundamentally differs from previous ones [7,19,44] in that the substitution rule in its standard format is derivable – this is key to our completeness result. §2 also presents an interpretation of the calculus via symmetric monoidal closed (a.k.a. autonomous) categories together with graded exponential comonads [19,30]. It then proves soundness of the aforementioned equational system w.r.t. this interpretation. §3 extends §2 to the $\mathcal{V}$-equational setting. Specifically, it equips our graded λ-calculus with a $\mathcal{V}$-equational system and shows how to interpret it via autonomous categories enriched over generalised metric spaces together with Lipschitz exponential comonads. It also shows that the $\mathcal{V}$-equational system is sound and complete w.r.t. this interpretation (Theorem 3.13). This result is highly generic and covers metric equations, classical (in)equations and ultra-metric and fuzzy variants. To the best of our knowledge this completeness result even for the basic case of classical equations is new. §4 details the aforementioned canonical construction of Lipschitz exponential comonads and §5 uses it as basis to provide metric higher-order models of both timed and probabilistic computation. In the former case the model that we canonically obtain is based on the category of metric spaces and non-expansive maps with the underlying Lipschitz comonad being that of dilations [30]. In the latter case the model is based on the category of Banach spaces and short linear maps with the underlying Lipschitz comonad arising from a process of symmetrisation well-known in linear algebra [6,8]. We assume basic knowledge of (enriched) category theory.

Related work. The need for quantitative notions of program equivalence has been explored in several concrete computational paradigms. This is the case for example of [47], [29], and [9,10] which introduce metric reasoning mechanisms for differential privacy, quantum, and probabilistic computation respectively. Other works take a more general perspective. For example on the side of universal algebra there has been great progress on the closely related topic of quantitative algebra, with focus typically on metric equations and inequations, see for example [39,40,48,2]. In fact, one case with a particularly interesting connection to ours is [11]: it explores a notion of quantitative equality with graded modalities and studies a corresponding *algebraic semantics* via Lawvere's doctrines. Our target is, however, λ-calculus. This sets us apart from these approaches, and in this regard positions us closer to the quantitative approaches targetting λ-calculi such as [20] and [21] which use the notion of a quantale to introduce quantitative counterparts of applicative (bi)similarity and rewriting systems respectively. Another example is [46] which studies quantitative semantics of simply-typed λ-calculi based on a generalisation of logical relations.

2 A graded λ-calculus and its interpretation

2.1 The calculus

We start by presenting our graded λ-calculus. In a nutshell, it is a graded extension of the linear-non-linear λ-calculus in [4,5] and can be seen as a term assignment system for a graded version of *intuitionistic linear logic*. Aside from the use of grades, the main difference with [4,5] is the use of a shuffling mechanism [50] that allows to refer to a λ-term's denotation unambiguously (more details below).

Types. As usual with graded modal types [23,19,44], we fix a semiring $\mathcal{R} = (R, 0, 1, +, \cdot)$ of 'resource quantities'. We then fix a set G of ground types and consider the following grammar of types:

$$\mathbb{A} ::= X \mid \mathbb{I} \mid \mathbb{A} \otimes \mathbb{A} \mid \mathbb{A} \multimap \mathbb{A} \mid !_r\, \mathbb{A} \qquad\qquad (X \in G, r \in \mathcal{R}).$$

Elements of R will be called *grades*. The grade r associated with a modal type $!_r\, \mathbb{A}$ intuitively represents *how much* of a resource we possess. For example, in the case of $\mathcal{R}$ being the semiring of natural numbers r may be regarded as the number of times a resource can be used before depletion.

Contexts and shuffles. We use Greek uppercase letters $\Gamma, \Delta, E, \dots$ to denote typing contexts, *i.e.* lists of typed variables $x_1 : \mathbb{A}_1, \dots, x_n : \mathbb{A}_n$ such that each x_i occurs at most once. As already mentioned, we will also use the notion of a shuffle: a permutation of typed variables in a context sequence $\Gamma_1, \dots, \Gamma_n$ such that for all $i \leq n$ the relative order of the variables in Γ_i is preserved [50]. For example, if $\Gamma_1 = x : \mathbb{A}, y : \mathbb{B}$ and $\Gamma_2 = z : \mathbb{C}$ then $z : \mathbb{C}, x : \mathbb{A}, y : \mathbb{B}$ is a shuffle but $y : \mathbb{B}, x : \mathbb{A}, z : \mathbb{C}$ is *not*, because we changed the order in which x and y appear in Γ_1. We denote by $\mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$ the set of shuffles on $\Gamma_1, \dots, \Gamma_n$. Shuffles will be used to build a graded λ-calculus where the exchange rule is admissible and at the same time each judgement $\Gamma \rhd v : \mathbb{A}$ has a unique derivation (Theorem 2.3). This will allow us to refer to a judgement's denotation $[\![\Gamma \rhd v : \mathbb{A}]\!]$ unambiguously.

Terms. Fix a set Σ of sorted operation symbols $f : \mathbb{A}_1, \dots, \mathbb{A}_n \to \mathbb{A}$ with $n \geq 1$. The term formation rules of the graded calculus are listed in Figure 1. By convention all contexts involved in the premisses of any of the listed rules are mutually disjoint. This entails for instance that in ($\otimes_\mathbf{e}$) neither x nor y can occur in Γ and analogously for ($!_{\mathbf{n+m}}$). The rules above the dotted line are standard and in correspondence to the natural deduction rules of exponential-free intuitionistic linear logic; we omit here their explanation. As for the others, the promotion rule ($!_\mathbf{i}$) allows the use of a term 'r-times' by intuitively binding all variables $x_i : !_{s_i}\, \mathbb{A}_i$ in its context to terms v_i whose type $!_{r \cdot s_i}\, \mathbb{A}_i$ is graded by the 'r-multiple' of s_i. The dereliction rule ($!_\mathbf{e}$) connects the modal typing system to the linear one, in particular it makes explicit that terms with linear types must be used exactly once. This is essential *e.g.* for using terms whose type is linear multiples times. Take for example the semiring of natural numbers and a sorted operation symbol $f : \mathbb{A} \to \mathbb{A}$. A call to f that is usable precisely 'r-times' is given by the judgement $y : !_r\, \mathbb{A} \rhd \mathtt{pr}_{(r,[1])}\, y\, \mathtt{fr}\, x.\, f(\mathtt{dr}\, x) : !_r\, \mathbb{A}$. Finally rules ($!_\mathbf{0}$) and ($!_{\mathbf{n+m}}$) correspond respectively to graded versions of weakening and contraction. They can be seen intuitively as discard and copy operations where in the latter case variables x and y are bound to the object v being copied.

Remark 2.1 When we instantiate $\mathcal{R}$ to the trivial semiring $(\{\infty\}, \infty, \infty, +, \cdot)$, the rules in Figure 1 are the ones presented in [4] modulo the shuffling mechanism.

$$\frac{\Gamma_i \rhd v_i : \mathbb{A}_i \quad f : \mathbb{A}_1, \dots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)}{E \rhd f(v_1, \dots, v_n) : \mathbb{A}} \ (\mathbf{ax}) \qquad \frac{}{x : \mathbb{A} \rhd x : \mathbb{A}} \ (\mathbf{hp})$$

$$\frac{}{- \rhd * : \mathbb{I}} \ (\mathbb{I}_\mathbf{i}) \qquad \frac{\Gamma \rhd v : \mathbb{I} \quad \Delta \rhd w : \mathbb{A} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v \ \mathtt{to} \ * .\ w : \mathbb{A}} \ (\mathbb{I}_\mathbf{e})$$

$$\frac{\Gamma \rhd v : \mathbb{A} \quad \Delta \rhd w : \mathbb{B} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v \otimes w : \mathbb{A} \otimes \mathbb{B}} \ (\otimes_\mathbf{i}) \qquad \frac{\Gamma \rhd v : \mathbb{A} \otimes \mathbb{B} \quad \Delta, x : \mathbb{A}, y : \mathbb{B} \rhd w : \mathbb{C} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd \mathtt{pm}\ v \ \mathtt{to}\ x \otimes y.\ w : \mathbb{C}} \ (\otimes_\mathbf{e})$$

$$\frac{\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}}{\Gamma \rhd \lambda x : \mathbb{A}.\, v : \mathbb{A} \multimap \mathbb{B}} \ (\multimap_\mathbf{i}) \qquad \frac{\Gamma \rhd v : \mathbb{A} \multimap \mathbb{B} \quad \Delta \rhd w : \mathbb{A} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd v\, w : \mathbb{B}} \ (\multimap_\mathbf{e})$$

. . . .

$$\frac{\Gamma_i \rhd v_i : !_{r \cdot s_i} \mathbb{A}_i \quad x_1 : !_{s_1} \mathbb{A}_1, \dots, x_n : !_{s_n} \mathbb{A}_n \rhd u : \mathbb{A} \quad E \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)}{E \rhd \mathtt{pr}_{(r,[s_1,\dots,s_n])}\ v_1, \dots, v_n \ \mathtt{fr}\ x_1, \dots, x_n.\ u : !_r \mathbb{A}} \ (!_\mathbf{i}) \qquad \frac{\Gamma \rhd v : !_1 \mathbb{A}}{\Gamma \rhd \mathtt{dr}\, v : \mathbb{A}} \ (!_\mathbf{e})$$

$$\frac{\Gamma \rhd v : !_0 \mathbb{A} \quad \Delta \rhd u : \mathbb{B} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd \mathtt{ds}\, v.\ u : \mathbb{B}} \ (!_\mathbf{0}) \qquad \frac{\Gamma \rhd v : !_{n+m} \mathbb{A} \quad \Delta, x : !_n \mathbb{A}, y : !_m \mathbb{A} \rhd u : \mathbb{B} \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{E \rhd \mathtt{cp}_{(n,m)}\ v \ \mathtt{to}\ x, y.\ u : \mathbb{B}} \ (!_\mathbf{n+m})$$

Fig. 1. Term formation rules of graded λ-calculus.

Properties. Our calculus has several desirable properties (Theorem 2.3 and Lemma 2.4), including the aforementioned fact that all judgements have a unique derivation. We start by presenting auxiliary notations. Given a context Γ we will use $te(\Gamma)$ to denote Γ with all types erased. Additionally, for contexts Γ and Γ' we will use notation $\Gamma \simeq_\pi \Gamma'$ to state that Γ is a permutation of Γ'. We will also use an analogous notation for non-repetitive lists of untyped variables $te(\Gamma)$. We will often abbreviate a judgement $\Gamma \rhd v : \mathbb{A}$ into $\Gamma \rhd v$ or even just v if no ambiguities arise. Finally, we will often denote a list of terms $v_1, \dots, v_n$ simply by $\boldsymbol{v}$ and analogously for lists of variables.

Proposition 2.2 *Let us consider two lists of contexts $\Gamma_1, \dots, \Gamma_n$ and $\Gamma'_1, \dots, \Gamma'_n$, contexts E and E', and suppose that $E \in \mathrm{Sf}(\Gamma_1; \dots; \Gamma_n)$, $E' \in \mathrm{Sf}(\Gamma'_1; \dots; \Gamma'_n)$. Then the following clauses hold:*

(i) *if $te(\Gamma_i) \simeq_\pi te(\Gamma'_i)$ for all $i \leq n$ then $te(E) \simeq_\pi te(E')$;*
(ii) *if $\Gamma_i \simeq_\pi \Gamma'_i$ for all $i \leq n$ then $E \simeq_\pi E'$;*
(iii) *if $E \simeq_\pi E'$ and $te(\Gamma_i) \simeq_\pi te(\Gamma'_i)$ for some $i \leq n$ then $\Gamma_i \simeq_\pi \Gamma'_i$;*
(iv) *if $E = E'$ and $te(\Gamma_i) \simeq_\pi te(\Gamma'_i)$ for some $i \leq n$ then $\Gamma_i = \Gamma'_i$.*

Theorem 2.3 *Graded λ-calculus has the following properties:*

(i) *for all judgements $\Gamma \rhd v$ and $\Gamma' \rhd v$ we have $te(\Gamma) \simeq_\pi te(\Gamma')$;*
(ii) *additionally if $\Gamma \rhd v : \mathbb{A}$, $\Gamma' \rhd v : \mathbb{A}'$, and $\Gamma \simeq_\pi \Gamma'$ then $\mathbb{A}$ must be equal to $\mathbb{A}'$;*
(iii) *all judgements $\Gamma \rhd v : \mathbb{A}$ have a unique derivation.*

Proof. The first clause follows straightforwardly from induction over the derivation system (Figure 1) and the first clause of Proposition 2.2. The second clause follows from induction over the derivation system, the first clause, the second and third clauses of Proposition 2.2, the grade annotations in term constructs, and the type annotation in the λ-construct. The third clause follows from induction over the derivation system, the second clause, the shuffling mechanism, and the fourth clause of Proposition 2.2. ☐

Substitution is defined in the expected way and as usual uses α-equivalence to avoid capturing free variables. In our setting such captures arise from the rules $(\multimap_{\mathbf{i}})$, $(\otimes_{\mathbf{e}})$, $(!_{\mathbf{i}})$, and $(!_{\mathbf{n+m}})$.

Lemma 2.4 (Exchange and Substitution) *For every judgement* $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}$ *we can derive* $\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd v : \mathbb{C}$. *For all judgements* $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$ *and* $\Delta \rhd w : \mathbb{A}$ *we can derive* $\Gamma, \Delta \rhd v[w/x] : \mathbb{B}$.

Proof. As usual the exchange property follows from induction over the derivation system in Figure 1. The substitution property follows from the exchange property, the fact that x occurs at most once in the term v, and from induction over the judgement derivation $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$. ☐

The substitution property proved in Lemma 2.4 generalises to iterated substitution. More specifically, given $\Gamma, x_1 : \mathbb{A}_1, \ldots, x_n : \mathbb{A}_n \rhd v : \mathbb{B}$ and $\Delta_i \rhd w_i : \mathbb{A}_i$ $(i \leq n)$ with all contexts involved pairwise disjoint one easily derives $\Gamma, \Delta_1, \ldots, \Delta_n \rhd v[w_1/x_1] \ldots [w_n/x_n] : \mathbb{B}$. Additionally it is straightforward to prove that, by virtue of all contexts being pairwise disjoint, the order in which the sequence of substitutions occurs is irrelevant. For this reason we will often abbreviate $v[w_1/x_1] \ldots [w_n/x_n]$ simply to $v[\boldsymbol{w}/\boldsymbol{x}]$ or $v[w_1/x_1, \ldots, w_n/x_n]$.

Remark 2.5 The promotion rule $(!_{\mathbf{i}})$ of our graded calculus differs from the promotion rule of previous calculi with graded modalities [7,19]. Let us explain this distinction and justify it. Let $\boldsymbol{s}$ denote a list of grades $s_1, \ldots, s_n$ and $r \cdot \boldsymbol{s}$ denote the list of grades $r \cdot s_1, \ldots, r \cdot s_n$. If we write $!_{\boldsymbol{s}}\, \Gamma$ to say that the type of every variable x_i in Γ is of the form $!_{s_i}\, \mathbb{A}_i$, then for every judgement $!_{\boldsymbol{s}}\, \Gamma \rhd v : \mathbb{A}$ with $te(\Gamma) = x_1, \ldots, x_n$ we can derive $!_{r \cdot \boldsymbol{s}}\, \Gamma \rhd \mathtt{pr}_{(r,\boldsymbol{s})}\, \boldsymbol{x}\ \mathtt{fr}\ \boldsymbol{y}.\, v[\boldsymbol{y}/\boldsymbol{x}] : !_r\, \mathbb{A}$ – we abbreviate the latter term simply to $!_r\, v$. The following rule is then admissible in our calculus:

$$\frac{!_{\boldsymbol{s}}\, \Gamma \rhd v : \mathbb{A}}{!_{r \cdot \boldsymbol{s}}\, \Gamma \rhd\, !_r\, v : !_r\, \mathbb{A}}$$

A rule with the same structural format is added *natively* to the calculi in [7,19] and is the counterpart to our promotion rule $(!_{\mathbf{i}})$. The former however breaks the substitution property stated in Lemma 2.4 (details available in [4, page 10]). This would hinder the development of our equational system and associated completeness result and justifies the slightly more complicated rule $(!_{\mathbf{i}})$.

Equational system. Figure 2 presents the equational schema of graded λ-calculus. As usual, we omit the typing information of the equations-in-context listed in Figure 2 which can be recovered uniquely up to permutations. The symbols $(:)$ and $(+\!\!+)$ denote usual operations on lists namely cons and concatenation. Note as well the division of the equational schema into different sections referring to specific categorical machinery. This is to attach a semantic intuition to the equations and to foreshadow the categorical structures that will be used later on to interpret graded λ-calculus. The equations concerning the monoidal structure and the closed structure were already discussed elsewhere (*e.g.* [4,13]). The equations concerning commuting conversions enforce the fact that certain expressions differing in scope such as $(\mathtt{ds}\, v.\, u) \otimes w$ and $\mathtt{ds}\, v.\, (u \otimes w)$ are intended to have the same meaning.

Next, in the axiomatisation of the comonadic structure, the first and second equations are respectively β and η equations and embody the counit laws associated to the underlying graded comonad. The third equation states that the inner promotion (on the left-hand side) can be pushed-forward to w but with the factor r_1 discarded as a result from not being bound to variable a anymore. This equation embodies the associativity law of the underlying graded comonad. Observe that for these three equations to be well-defined the reduct $(R, 1, \cdot)$ in the semiring $\mathcal{R}$ needs to be a monoid (which we assumed previously). The fourth equation tells that the order in which terms $\boldsymbol{v}$ appear in a promotion $\mathtt{pr}_{(r,\boldsymbol{s})}\, \boldsymbol{v}\, \mathtt{fr}\, \boldsymbol{x}.\, u$ is irrelevant, which fact embodies the symmetry of the graded comonad.

The discard (*i.e.* weakening) and copy (*i.e.* contraction) operations suggest a (graded) commutative comonoidal structure, which is reflected in the four corresponding equations in Figure 2. This time, these equations force the reduct $(R, 0, +)$ in the semiring $\mathcal{R}$ to be a commutative monoid (which indeed we also assumed previously). In the axiomatisation of the interaction between the underlying comonoid and comonad, the first two equations can be seen as a mechanism for shifting term complexity between the discard and promotion expressions (this is noticeable by looking at the grade annotations in the promotions, when present). They may equally well be regarded respectively as β and η-equations whose corresponding reduction simplifies the promotion expression. Semantically they reflect the naturality of the discard operation, that the latter is a graded version of a coalgebra morphism, and that the comonad's comultiplication is a comonoid morphism (we formally detail this later on). Note as well that these equations force 0 to be an absorbing element of the monoid operation $(\cdot)$ in the semiring $\mathcal{R}$ (which indeed we assumed previously). The last two equations follow a reasoning analogous to the previous two, and force $(\cdot)$ to distribute over $(+)$ both on the left and the right (which we also assumed). The equations described thus entail that $\mathcal{R}$ has a semiring structure as previously postulated.

Remark 2.6 This equational schema is a graded generalisation of the one presented in [4]. In fact, for the particular case of the singleton semiring $\mathcal{R} = \{\{\infty\}, \infty, \infty, +, \cdot\}$ our equations collapse to those in [4] except for the equation about the comonad's symmetry which is absent from *op. cit.*

2.2 *The interpretation*

In this subsection we present an interpretation of the graded calculus detailed above. The interpretation uses the categorical machinery suggested in [19,30,44] to interpret previous graded calculi. We also prove that the equational schema in Figure 2 is sound w.r.t. this interpretation.

We start by recalling preliminary categorical notions and some conventions concerning symmetric monoidal closed (*i.e.* autonomous) categories. Given one such category C and for a list of C-objects $X_1, \dots, X_n$ we write $X_1 \otimes \cdots \otimes X_n$ for the n-tensor $(\dots(X_1 \otimes X_2) \otimes \dots) \otimes X_n$ and similarly for morphisms. For all C-objects X, Y, Z, $\gamma : X \otimes Y \to Y \otimes X$ denotes the symmetry morphism, $\lambda : I \otimes X \to X$ the left unitor, $\mathrm{app} : (X \multimap Y) \otimes X \to Y$ the application morphism, and $\alpha : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ the left associator. For all C-morphisms $f : X \otimes Y \to Z$ we denote the corresponding curried version by $\overline{f} : X \to (Y \multimap Z)$. We will frequently omit subscripts in natural transformations. For a monoidal functor $F : \mathsf{C} \to \mathsf{C}$ we denote by $\phi : I \to FI$ and $\phi_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ the corresponding monoidal operations. Similarly given C-objects $X_1, \dots, X_n$ we denote by $\phi_{X_1,\dots,X_n} : FX_1 \otimes \cdots \otimes FX_n \to F(X_1 \otimes \cdots \otimes X_n)$ the morphism defined recursively on the size of n by:

$$\phi_{-} = \phi \qquad \phi_X = \mathsf{id} \qquad \phi_{X_1,\dots,X_n,X_{n+1}} = \phi_{(X_1 \otimes \cdots \otimes X_n),X_{n+1}} \cdot (\phi_{X_1,\dots,X_n} \otimes \mathsf{id}).$$

In the presence of several monoidal functors F, G, we denote their respective monoidal operations by ϕ^F, ϕ^G.

We now set the ground for the notion of a graded exponential comonad, explored for example in [19,30,44] and standardly used for interpreting graded modal types. Note first that a semiring $\mathcal{R} = (R, 0, 1, +, \cdot)$ has two (interacting) monoidal structures: $(R, 0, +)$ (which is commutative) and $(R, 1, \cdot)$ (which need not be). The category $[\mathsf{C}, \mathsf{C}]$ of endofunctors and natural transformations also has two monoidal structures, specifically $([\mathsf{C}, \mathsf{C}], I, \otimes)$ (where I designates to constant functor to the unit) and $([\mathsf{C}, \mathsf{C}], \mathrm{Id}, \circ)$. The category $\mathsf{Mon}[\mathsf{C}, \mathsf{C}]$ (resp. $\mathsf{SymMon}[\mathsf{C}, \mathsf{C}]$) of *monoidal* (resp. *symmetric monoidal*) endofunctors and *monoidal* natural transformations inherits these two monoidal structures from $[\mathsf{C}, \mathsf{C}]$. The semantics of our graded λ-calculus relies on a 'representation' of $\mathcal{R}$ in C using these two structures, as detailed below.

Definition 2.7 An $\mathcal{R}$-*graded comonad* over a (not necessarily monoidal) category C is an oplax monoidal functor $D : (R, 1, \cdot) \to ([\mathsf{C}, \mathsf{C}], \mathrm{Id}, \circ)$. Similarly, an $\mathcal{R}$-graded *monoidal* comonad is an oplax monoidal functor $D : (R, 1, \cdot) \to (\mathsf{Mon}[\mathsf{C}, \mathsf{C}], \mathrm{Id}, \circ)$, and an $\mathcal{R}$-graded *symmetric monoidal* comonad is an oplax monoidal functor $D : (R, 1, \cdot) \to (\mathsf{SymMon}[\mathsf{C}, \mathsf{C}], \mathrm{Id}, \circ)$. Concretely, an $\mathcal{R}$-graded comonad is a triple

Monoidal structure	Closed structure
$\mathtt{pm}\ v \otimes w\ \mathtt{to}\ x \otimes y.\ u = u[v/x, w/y]$	$(\lambda x : \mathbb{A}.\ v)\ w = v[w/x]$
$\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ u[x \otimes y/z] = u[v/z]$	$\lambda x : \mathbb{A}.\ v\ x = v$
$* \ \mathtt{to}\ *.\ v = v$	
$v\ \mathtt{to}\ *.\ w[*/z] = w[v/z]$	

Symmetric comonadic structure

$$
\begin{aligned}
\mathtt{dr}\ \mathtt{pr}_{(1,\boldsymbol{s})}\ \boldsymbol{v}\ \mathtt{fr}\ \boldsymbol{x}.\ u &= u[\boldsymbol{v}/\boldsymbol{x}] \\
\mathtt{pr}_{(r,[1])}\ z\ \mathtt{fr}\ x.\ \mathtt{dr}\ x &= z \\
\mathtt{pr}_{(r_1,r_2:\boldsymbol{r})}\ (\mathtt{pr}_{(r_1\cdot r_2,\boldsymbol{s})}\ \boldsymbol{x}\ \mathtt{fr}\ \boldsymbol{y}.\ v), \boldsymbol{z}\ \mathtt{fr}\ a, \boldsymbol{a}.\ w &= \mathtt{pr}_{(r_1,(r_2\cdot\boldsymbol{s})+\!\!+\boldsymbol{r})}\ \boldsymbol{x}, \boldsymbol{z}\ \mathtt{fr}\ \boldsymbol{c}, \boldsymbol{a}.\ w[\mathtt{pr}_{(r_2,\boldsymbol{s})}\ \boldsymbol{c}\ \mathtt{fr}\ \boldsymbol{y}.\ v/a] \\
\mathtt{pr}_{(r,\boldsymbol{s_1}+\!\!+[r_1,r_2]+\!\!+\boldsymbol{s_2})}\ \boldsymbol{v}_1, w_1, w_2, \boldsymbol{v}_2\ \mathtt{fr}\ \boldsymbol{x}_1, y_1, y_2, \boldsymbol{x}_2.\ u &= \mathtt{pr}_{(r,\boldsymbol{s_1}+\!\!+[r_2,r_1]+\!\!+\boldsymbol{s_2})}\ \boldsymbol{v}_1, w_2, w_1, \boldsymbol{v}_2\ \mathtt{fr}\ \boldsymbol{x}_1, y_2, y_1, \boldsymbol{x}_2.\ u
\end{aligned}
$$

Commutative comonoid structure

$$
\begin{aligned}
\mathtt{cp}_{(0,n)}\ v\ \mathtt{to}\ x, y.\ \mathtt{ds}\ x.\ u &= u[v/y] \\
\mathtt{cp}_{(n,0)}\ v\ \mathtt{to}\ x, y.\ \mathtt{ds}\ y.\ u &= u[v/x] \\
\mathtt{cp}_{(n+m,o)}\ v\ \mathtt{to}\ x, y.\ \mathtt{cp}_{(n,m)}\ x\ \mathtt{to}\ a, b.\ u &= \mathtt{cp}_{(n,m+o)}\ v\ \mathtt{to}\ a, c.\ \mathtt{cp}_{(m,o)}\ c\ \mathtt{to}\ b, y.\ u \\
\mathtt{cp}_{(n,m)}\ v\ \mathtt{to}\ x, y.\ u &= \mathtt{cp}_{(m,n)}\ v\ \mathtt{to}\ y, x.\ u
\end{aligned}
$$

Interaction between comonoid and comonad

$$
\begin{aligned}
\mathtt{ds}\ \mathtt{pr}_{(0,\boldsymbol{s})}\ \boldsymbol{v}\ \mathtt{fr}\ \boldsymbol{x}.\ w.\ u &= \mathtt{ds}\ v_1.\ \dots\ \mathtt{ds}\ v_n.\ u \\
\mathtt{pr}_{(r,0:\boldsymbol{s})}\ v, \boldsymbol{v}\ \mathtt{fr}\ x, \boldsymbol{x}.\ \mathtt{ds}\ x.\ u &= \mathtt{ds}\ v.\ \mathtt{pr}_{(r,\boldsymbol{s})}\ \boldsymbol{v}\ \mathtt{fr}\ \boldsymbol{x}.\ u \\
\mathtt{cp}_{(n,m)}\ \mathtt{pr}_{(n+m,[s_1,\dots,s_k])}\ \boldsymbol{v}\ \mathtt{fr}\ \boldsymbol{x}.\ w\ \mathtt{to}\ y, z.\ u &= \mathtt{cp}_{(n\cdot s_1,m\cdot s_1)}\ v_1\ \mathtt{to}\ a_1, b_1.\ \dots\ \mathtt{cp}_{(n\cdot s_k,m\cdot s_k)}\ v_k\ \mathtt{to}\ a_k, b_k. \\
&\qquad u[\mathtt{pr}_{(n,[s_1,\dots,s_k])}\ \boldsymbol{a}\ \mathtt{fr}\ \boldsymbol{x}.\ w/y, \mathtt{pr}_{(m,[s_1,\dots,s_k])}\ \boldsymbol{b}\ \mathtt{fr}\ \boldsymbol{x}.\ w/z] \\
\mathtt{pr}_{(r,(n+m):\boldsymbol{s})}\ v, \boldsymbol{v}\ \mathtt{fr}\ z, \boldsymbol{z}.\ \mathtt{cp}_{(n,m)}\ z\ \mathtt{to}\ x, y.\ u &= \mathtt{cp}_{(r\cdot n,r\cdot m)}\ v\ \mathtt{to}\ a, b.\ \mathtt{pr}_{(r,n:m:\boldsymbol{s})} a, b, \boldsymbol{v}\ \mathtt{fr}\ x, y, \boldsymbol{z}.\ u
\end{aligned}
$$

Commuting conversions

$$
\begin{aligned}
u[v\ \mathtt{to}\ *.\ w/z] &= v\ \mathtt{to}\ *.\ u[w/z] \\
u[\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ w/z] &= \mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ u[w/z] \\
u[\mathtt{ds}\ v.\ w/z] &= \mathtt{ds}\ v.\ u[w/z] \\
u[\mathtt{cp}_{(n,m)}\ v\ \mathtt{to}\ x, y.\ w/z] &= \mathtt{cp}_{(n,m)}\ v\ \mathtt{to}\ x, y.\ u[w/z]
\end{aligned}
$$

Fig. 2. Equational schema of graded λ-calculus.

$(D_{(-)} : R \to [\mathsf{C}, \mathsf{C}], \epsilon : D_1 \to \mathrm{Id}, \delta^{m,n} : D_{m\cdot n} \to D_m D_n)$ that makes the following diagrams commute

$$
\begin{array}{ccc}
D_s & \xrightarrow{\delta^{s,1}} & D_s D_1 \\
{\scriptstyle \delta^{1,s}}\downarrow & \searrow^{=} & \downarrow{\scriptstyle D_s\epsilon} \\
D_1 D_s & \xrightarrow[\epsilon_{D_s}]{} & D_s
\end{array}
\qquad\qquad
\begin{array}{ccc}
D_{s_1\cdot s_2\cdot s_3} & \xrightarrow{\delta^{s_1,s_2\cdot s_3}} & D_{s_1} D_{s_2\cdot s_3} \\
{\scriptstyle \delta^{s_1\cdot s_2,s_3}}\downarrow & & \downarrow{\scriptstyle D_{s_1}\delta^{s_2,s_3}} \\
D_{s_1\cdot s_2} D_{s_3} & \xrightarrow[\delta^{s_1,s_2}{}_{D_{s_3}}]{} & D_{s_1} D_{s_2} D_{s_3}
\end{array}
\qquad (2)
$$

and similarly for an $\mathcal{R}$-graded monoidal and symmetric monoidal comonad.

Definition 2.8 An $\mathcal{R}$-*graded exponential comonad* is an $\mathcal{R}$-graded symmetric monoidal comonad D :

$(R, 1, \cdot) \to (\mathsf{SymMon}[\mathsf{C}, \mathsf{C}], \mathrm{Id}, \circ)$ that satisfies the following additional properties:

(i) D is an oplax symmetric monoidal functor $D : (R, 0, +) \to (\mathsf{SymMon}[\mathsf{C}, \mathsf{C}], I, \otimes)$. In other words, we have monoidal natural transformations $e : D_0 \to I$ and $d^{m,n} : D_{m+n} \to D_m \otimes D_n$ making the analogues of (2) for the monoidal structure $([\mathsf{C}, \mathsf{C}], I, \otimes)$ commute. Note that since $(R, 0, +)$ is commutative and D is symmetric the diagram below commutes as well.

$$\begin{array}{ccc} D_{m+n} & = & D_{n+m} \\ {\scriptstyle d^{m,n}}\downarrow & & \downarrow{\scriptstyle d^{n,m}} \\ D_m \otimes D_n & \xrightarrow[\gamma]{} & D_n \otimes D_m \end{array}$$

This equips every C-object with the structure of a graded commutative comonoid [18].

(ii) The two oplax monoidal structures of D interact as specified by the diagrams below (where the transformations ϕ^{D_n} and $\phi^{D_s}_{-,-}$ are available by virtue of the typing of D).

$$\begin{array}{ccc} D_{n\cdot 0} & \xrightarrow{\delta^{n,0}} & D_n D_0 \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle D_n e} \\ I & \xrightarrow[\phi^{D_n}]{} & D_n I \end{array} \qquad \begin{array}{ccc} D_{0\cdot s} & \xrightarrow{\delta^{0,s}} & D_0 D_s \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e_{D_s}} \\ I & = & I \end{array} \qquad \begin{array}{ccc} D_{(n+m)\cdot s} & \xrightarrow{\delta^{n+m,s}} & D_{n+m} D_s \\ {\scriptstyle d^{n\cdot s, m\cdot s}}\downarrow & & \downarrow{\scriptstyle d^{n,m}_{D_s}} \\ D_{n\cdot s} \otimes D_{m\cdot s} & \xrightarrow[\delta^{n,s} \otimes \delta^{m,s}]{} & D_n D_s \otimes D_m D_s \end{array}$$

$$\begin{array}{ccccc} D_{s\cdot(n+m)} & & \xrightarrow{\delta^{s,(n+m)}} & & D_s D_{n+m} \\ {\scriptstyle d^{(s\cdot n)+(s\cdot m)}}\downarrow & & & & \downarrow{\scriptstyle D_s d^{n,m}} \\ D_{s\cdot n} \otimes D_{s\cdot m} & \xrightarrow[\delta^{s,n} \otimes \delta^{s,m}]{} & D_s D_n \otimes D_s D_m & \xrightarrow[\phi^{D_s}_{D_n, D_m}]{} & D_s(D_n \otimes D_m) \end{array}$$

We now show how to interpret graded λ-calculus in an autonomous category C equipped with a graded exponential comonad D. For every ground type $X \in G$ we fix an interpretation $[\![X]\!]$ as a C-object and interpret the type structure inductively in the usual way. Modal types are interpreted via the underlying graded comonad, specifically we set $[\![!_r\, \mathbb{A}]\!] = D_r[\![\mathbb{A}]\!]$. Given a non-empty context $\Gamma = \Gamma', x : \mathbb{A}$, its interpretation is defined by $[\![\Gamma', x : \mathbb{A}]\!] = [\![\Gamma']\!] \otimes [\![\mathbb{A}]\!]$ if Γ' is non-empty and $[\![\Gamma', x : \mathbb{A}]\!] = [\![\mathbb{A}]\!]$ otherwise. The empty context is interpreted as $[\![-]\!] = I$ where I is the unit of $\otimes$ in C. We will also need some 'housekeeping' morphisms to handle interactions between context interpretation and the symmetric monoidal structure of C. Given contexts $\Gamma_1, \ldots, \Gamma_n$ we denote by $\mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} : [\![\Gamma_1, \ldots, \Gamma_n]\!] \to [\![\Gamma_1]\!] \otimes \cdots \otimes [\![\Gamma_n]\!]$ the morphism that splits $[\![\Gamma_1, \ldots, \Gamma_n]\!]$ into $[\![\Gamma_1]\!] \otimes \cdots \otimes [\![\Gamma_n]\!]$, and by $\mathsf{jn}_{\Gamma_1;\ldots;\Gamma_n}$ the corresponding inverse. Given a context $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta$ we denote by $\mathsf{exch}_{\Gamma,\underline{x:\mathbb{A},y:\mathbb{B}},\Delta} : [\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta]\!] \to [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta]\!]$ the morphism corresponding to the permutation of the variable $x : \mathbb{A}$ with $y : \mathbb{B}$. Whenever convenient we will drop variable names in the subscripts of sp, jn, and exch. Given a context $E \in \mathrm{Sf}(\Gamma_1; \ldots; \Gamma_n)$ the morphism $\mathsf{sh}_E : [\![E]\!] \to [\![\Gamma_1, \ldots, \Gamma_n]\!]$ denotes the corresponding shuffling morphism. For every sorted operation $f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A} \in \Sigma$ we set $[\![f]\!] : [\![\mathbb{A}_1]\!] \otimes \cdots \otimes [\![\mathbb{A}_n]\!] \to [\![\mathbb{A}]\!]$ as a C-morphism. Finally we use the rules in Figure 3 to interpret judgements $\Gamma \rhd v : \mathbb{A}$ as C-morphisms via induction over the judgement derivation system in Figure 1.

The following lemma is standard and like in analogous contexts useful for proving the soundness theorem presented below.

Lemma 2.9 (Exchange and Substitution) *For all judgements $\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}$, $\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}$, and $\Delta \rhd w : \mathbb{A}$, the following equations hold.*

$$[\![\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \rhd v : \mathbb{C}]\!] = [\![\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \rhd v : \mathbb{C}]\!] \cdot \mathsf{exch}_{\Gamma,\underline{\mathbb{A},\mathbb{B}},\Delta}$$
$$[\![\Gamma, \Delta \rhd v[w/x] : \mathbb{B}]\!] = [\![\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}]\!] \cdot \mathsf{jn}_{\Gamma;\mathbb{A}} \cdot (\mathrm{id} \otimes [\![\Delta \rhd w : \mathbb{A}]\!]) \cdot \mathsf{sp}_{\Gamma;\Delta}$$

$$\frac{[\![\Gamma_i \rhd v_i : \mathbb{A}_i]\!] = h_i \quad f : \mathbb{A}_1, \ldots, \mathbb{A}_n \to \mathbb{A} \in \Sigma \quad E \in \mathrm{Sf}(\Gamma_1; \ldots; \Gamma_n)}{[\![E \rhd f(v_1, \ldots, v_n) : \mathbb{A}]\!] = [\![f]\!] \cdot (h_1 \otimes \cdots \otimes h_n) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_E} \qquad \frac{}{[\![x : \mathbb{A} \rhd x : \mathbb{A}]\!] = \mathsf{id}_{[\![\mathbb{A}]\!]}}$$

$$\frac{}{[\![- \rhd * : \mathbb{I}]\!] = \mathsf{id}_{[\![\mathbb{I}]\!]}} \qquad \frac{[\![\Gamma \rhd v : \mathbb{A} \otimes \mathbb{B}]\!] = g \quad [\![\Delta, x : \mathbb{A}, y : \mathbb{B} \rhd w : \mathbb{C}]\!] = h \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd \texttt{pm}\ v\ \texttt{to}\ x \otimes y.\ w : \mathbb{C}]\!] = h \cdot \mathsf{jn}_{\Delta;\mathbb{A};\mathbb{B}} \cdot \alpha \cdot \gamma \cdot (g \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma \rhd v : \mathbb{A}]\!] = g \quad [\![\Delta \rhd w : \mathbb{B}]\!] = h \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd v \otimes w : \mathbb{A} \otimes \mathbb{B}]\!] = (g \otimes h) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E} \qquad \frac{[\![\Gamma \rhd v : \mathbb{I}]\!] = g \quad [\![\Delta \rhd w : \mathbb{A}]\!] = h \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd v\ \texttt{to}\ *.\ w : \mathbb{A}]\!] = h \cdot \lambda \cdot (g \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma, x : \mathbb{A} \rhd v : \mathbb{B}]\!] = h}{[\![\Gamma \rhd \lambda x : \mathbb{A}.\, v : \mathbb{A} \multimap \mathbb{B}]\!] = \overline{(h \cdot \mathsf{jn}_{\Gamma;\mathbb{A}})}} \qquad \frac{[\![\Gamma \rhd v : \mathbb{A} \multimap \mathbb{B}]\!] = g \quad [\![\Delta \rhd w : \mathbb{A}]\!] = h \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd v\, w : \mathbb{B}]\!] = \mathrm{app} \cdot (g \otimes h) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

. . . .

$$\frac{[\![\Gamma \rhd v : !_1\, \mathbb{A}]\!] = h}{[\![\Gamma \rhd \texttt{dr}\, v : \mathbb{A}]\!] = \epsilon_{[\![\mathbb{A}]\!]} \cdot h} \qquad \frac{[\![\Gamma \rhd v : !_0\, \mathbb{A}]\!] = g \quad [\![\Delta \rhd w : \mathbb{B}]\!] = h \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd \texttt{ds}\, v.\ w : \mathbb{B}]\!] = h \cdot \lambda \cdot (e_{[\![\mathbb{A}]\!]} \otimes \mathsf{id}) \cdot (g \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma \rhd v : !_{n+m}\ \mathbb{A}]\!] = g \quad [\![\Delta, x : !_n\, \mathbb{A}, y : !_m\, \mathbb{A} \rhd u : \mathbb{B}]\!] = h \quad E \in \mathrm{Sf}(\Gamma; \Delta)}{[\![E \rhd \texttt{cp}_{(n,m)}\ v\ \texttt{to}\ x, y.\ u : \mathbb{B}]\!] = h \cdot \mathsf{jn}_{\Delta;\mathbb{A};\mathbb{A}} \cdot \alpha \cdot \gamma \cdot (d^{n,m}_{[\![\mathbb{A}]\!]} \otimes \mathsf{id}) \cdot (g \otimes \mathsf{id}) \cdot \mathsf{sp}_{\Gamma;\Delta} \cdot \mathsf{sh}_E}$$

$$\frac{[\![\Gamma_i \rhd v_i : !_{r \cdot s_i}\, \mathbb{A}_i]\!] = g_i \quad [\![x_1 : !_{s_1}\, \mathbb{A}_1, \ldots, x_n : !_{s_n}\, \mathbb{A}_n \rhd u : \mathbb{A}]\!] = h \quad E \in \mathrm{Sf}(\Gamma_1; \ldots; \Gamma_n)}{[\![E \rhd \texttt{pr}_{(r,\boldsymbol{s})}\ \boldsymbol{v}\ \texttt{fr}\ \boldsymbol{x}.\ u : !_r\, \mathbb{A}]\!] = D_r h \cdot D_r\, \mathsf{jn}_{\mathbb{A}_1;\ldots;\mathbb{A}_n} \cdot \phi^{D_r}_{[\![\mathbb{A}_1]\!],\ldots,[\![\mathbb{A}_n]\!]} \cdot (\delta^{r,s_1}_{[\![\mathbb{A}_1]\!]} \otimes \cdots \otimes \delta^{r,s_n}_{[\![\mathbb{A}_n]\!]}) \cdot (g_1 \otimes \cdots \otimes g_n) \cdot \mathsf{sp}_{\Gamma_1;\ldots;\Gamma_n} \cdot \mathsf{sh}_E}$$

Fig. 3. Judgement interpretation.

Theorem 2.10 (Soundness) *The equations presented in Figure 2 are sound w.r.t. judgement interpretation. More specifically if $\Gamma \rhd v = w : \mathbb{A}$ is one of the equations in Figure 2 then $[\![\Gamma \rhd v : \mathbb{A}]\!] = [\![\Gamma \rhd w : \mathbb{A}]\!]$.*

3 A complete $\mathcal{V}$-equational system for graded λ-calculus

We now present a $\mathcal{V}$-equational system for graded λ-calculus and prove its soundness and completeness.

3.1 The $\mathcal{V}$-equational system

We start by recalling from [13] the conditions imposed on $\mathcal{V}$ to obtain a well-behaved framework of $\mathcal{V}$-equations. We will then extend this framework to the graded seting. Let $\mathcal{V}$ denote a commutative and unital quantale, $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ the corresponding binary operation, and k the unit [45]. Consider now the two following definitions concerning ordered structures [22,25] (they will allow us to work with specified subsets of $\mathcal{V}$-equations chosen *e.g.* for computational reasons [13]).

Definition 3.1 Take a complete lattice L. For every $x, y \in L$ we say that y is *way-below* x (in symbols, $y \ll x$) if for every subset $X \subseteq L$ whenever $x \leq \bigvee X$ there exists a *finite* subset $A \subseteq X$ such that $y \leq \bigvee A$. The lattice L is called *continuous* iff for every $x \in L$,

$$x = \bigvee \{y \mid y \in L \text{ and } y \ll x\}$$

Definition 3.2 Let L be a complete lattice. A *basis* B of L is a subset $B \subseteq L$ such that for every $x \in L$ the set $B \cap \{y \mid y \in L \text{ and } y \ll x\}$ is directed and has x as the least upper bound.

We assume that the underlying lattice of $\mathcal{V}$ is continuous and has a basis $B \ni k$ closed under finite joins and multiplication. As alluded above, the continuity condition will allow us to work only with $\mathcal{V}$-equations whose label is in B. We also assume that $\mathcal{V}$ is *integral*, *i.e.* that the unit k is the top element of $\mathcal{V}$, a common assumption in quantale theory [3] that facilitates some of our results.

Example 3.3 The Boolean quantale $((\{0 \leq 1\}, \vee), \otimes := \wedge)$ is *finite* and thus continuous [22]. Since it is continuous, $\{0, 1\}$ itself is a basis for the quantale that satisfies the conditions above. For the metric quantale $(([0,\infty], \wedge), \otimes := +)$ (note that the order on this quantale is the opposite of the usual order on $[0,\infty]$), the way-below relation corresponds to the *strictly greater* relation with $\infty > \infty$, and a basis for the underlying lattice that satisfies the conditions above is the set of extended non-negative rational numbers. Other examples of quantales that satisfy the conditions above can be found in [13].

A $\mathcal{V}$-equation-in-context is an expression $\Gamma \rhd v =_q w : \mathbb{A}$ where $q \in B$ (the basis of $\mathcal{V}$), and $\Gamma \rhd v : \mathbb{A}$, $\Gamma \rhd w : \mathbb{A}$ are graded λ-terms. If $\mathcal{V}$ is the metric quantale we obtain metric equations-in-context and if $\mathcal{V}$ is the Boolean quantale we obtain inequations-in-context (where $v =_1 w$ corresponds to $v \leq w$). In this $\mathcal{V}$-equational setting a classical equation-in-context $v = w$ translates to $v =_k w \wedge w =_k v$. For example in the metric case $v = w \equiv v =_0 w \wedge w =_0 v$ and in the Boolean case $v = w \equiv v \leq w \wedge w \leq v$.

We can now move to the graded setting.

Definition 3.4 A *scalar multiplication* of a semiring $\mathcal{R}$ on a quantale $\mathcal{V}$ is a function $\bullet : R \times \mathcal{V} \to \mathcal{V}$ such that for each $k \in R$, the map $k \bullet - : \mathcal{V} \to \mathcal{V}$ preserves joins in $\mathcal{V}$.

The definition entails in particular that for all $v, v' \in \mathcal{V}$ if $v \geq v'$ then $k \bullet v \geq k \bullet v'$.

Definition 3.5 [Graded $\mathcal{V}\lambda$-theories] Consider a tuple (G, Σ) consisting of a set G of ground types and a set Σ of sorted operation symbols. A *graded* $\mathcal{V}\lambda$*-theory* $((G, \Sigma), Ax)$ is a triple such that Ax is a set of $\mathcal{V}$-equations-in-context between λ-terms built from (G, Σ).

The elements of Ax are called the *axioms* of the theory. Let $Th(Ax)$ be the smallest $\mathcal{V}$-indexed binary relation (the $\mathcal{V}$-equations) that contains Ax, the equational schema presented in Figure 2, and that is closed under the rules listed in Figure 4. We call the elements of $Th(Ax)$ the *theorems* of the theory. Intuitively the rules in Figure 4 above the first dotted line can be seen as a $\mathcal{V}$-generalisation of an equivalence relation (see [13] for a more detailed explanation). The other rules correspond to a $\mathcal{V}$-generalisation of compatibility. The rule concerning promotion is slightly different from the others in that it involves a k-factor $(k \bullet -)$ to reflect the fact that u (resp. u') becomes usable k-times. Finally, note that we can consider *symmetric* graded $\mathcal{V}\lambda$-theories by adding to the mix the rule,

$$\frac{v =_q w}{w =_q v}$$

This is desirable for example in the (ultra-)metric case but makes no sense if one wishes to work with inequations (graded inequational λ-theories collapse to graded equational ones under this rule).

3.2 Interpretation of $\mathcal{V}$-equations, soundness, and completeness

In this subsection we recall the interpretation of $\mathcal{V}$-equations in the setting of linear λ-calculus [13] and extend it to the graded case. The main idea is that we suitably enrich the interpretation structure in Definition 2.8 (an autonomous category equipped with a graded exponential comonad) so that the corresponding hom-sets become equipped with a 'generalised metric structure'. More technically the basis of enrichment is that of $\mathcal{V}$-categories [35,51,27,3], a concept which we recall below. We prove soundness and completeness of the previous $\mathcal{V}$-equational system w.r.t. this interpretation.

Definition 3.6 A $\mathcal{V}$-category is a pair (X, a) where X is a set and $a : X \times X \to \mathcal{V}$ is a function that satisfies $k \leq a(x, x)$ and $a(x, y) \otimes a(y, z) \leq a(x, z)$ for all $x, y, z \in X$. For two $\mathcal{V}$-categories (X, a) and (Y, b), a $\mathcal{V}$-functor $f : (X, a) \to (Y, b)$ is a function $f : X \to Y$ that satisfies the inequality $a(x, y) \leq b(f(x), f(y))$ for all $x, y \in X$.

$$\frac{}{v =_\top v}\ \textbf{(refl)} \qquad \frac{v =_{q_1} w \qquad w =_{q_2} u}{v =_{q_1 \otimes q_2} u}\ \textbf{(trans)} \qquad \frac{v =_{q_1} w \qquad q_2 \leq q_1}{v =_{q_2} w}\ \textbf{(weak)}$$

$$\frac{\forall q_2 \ll q_1.\ v =_{q_2} w}{v =_{q_1} w}\ \textbf{(arch)} \qquad \frac{\forall i \leq n.\ v =_{q_i} w}{v =_{\vee q_i} w}\ \textbf{(join)}$$

. . . .

$$\frac{\forall i \leq n.\ v_i =_{q_i} w_i}{f(v_1, \dots, v_n) =_{\otimes q_i} f(w_1, \dots, w_n)} \qquad \frac{v =_{q_1} w \qquad v' =_{q_2} w'}{\mathtt{pm}\ v\ \mathtt{to}\ x \otimes y.\ v' =_{q_1 \otimes q_2} \mathtt{pm}\ w\ \mathtt{to}\ x \otimes y.\ w'} \qquad \frac{v =_{q_1} w \qquad v' =_{q_2} w'}{v\ \mathtt{to}\ *.\ v' =_{q_1 \otimes q_2} w\ \mathtt{to}\ *.\ w'}$$

$$\frac{v =_{q_1} w \quad v' =_{q_2} w'}{v \otimes v' =_{q_1 \otimes q_2} w \otimes w'} \qquad \frac{v =_q w}{\lambda x : \mathbb{A}.\ v =_q \lambda x : \mathbb{A}.\ w} \qquad \frac{v =_{q_1} w \quad v' =_{q_2} w'}{v\, v' =_{q_1 \otimes q_2} w\, w'}$$

. . . .

$$\frac{v =_q v'}{\mathtt{dr}\, v =_q \mathtt{dr}\, v'} \qquad \frac{v =_{q_1} v' \quad w =_{q_2} w'}{\mathtt{cp}_{(n,m)} v\ \mathtt{to}\ x, y.\ w =_{q_1 \otimes q_2} \mathtt{cp}_{(n,m)} v'\ \mathtt{to}\ x, y.\ w'} \qquad \frac{v =_{q_1} v' \quad w =_{q_2} w'}{\mathtt{ds}\, v.\ w =_{q_1 \otimes q_2} \mathtt{ds}\, v'.\ w'}$$

$$\frac{\Gamma \triangleright v =_q w : \mathbb{A} \qquad \Delta \in \mathrm{perm}(\Gamma)}{\Delta \triangleright v =_q w : \mathbb{A}} \qquad \frac{\forall i \leq n.\ v_i =_{q_i} v'_i \qquad u =_{q'} u'}{\mathtt{pr}_{(r,s)}\ \boldsymbol{v}\ \mathtt{fr}\ \boldsymbol{x}.\ u =_{\otimes q_i \otimes (r \bullet q')} \mathtt{pr}_{(r,s)}\ \boldsymbol{v'}\ \mathtt{fr}\ \boldsymbol{x}.\ u'} \qquad \frac{v =_{q_1} w \qquad v' =_{q_2} w'}{v[v'/x] =_{q_1 \otimes q_2} w[w'/x]}$$

Fig. 4. $\mathcal{V}$-congruence rules.

$\mathcal{V}$-categories and $\mathcal{V}$-functors form a category which we denote by $\mathcal{V}$-Cat. A $\mathcal{V}$-category (X, a) is called *symmetric* if $a(x, y) = a(y, x)$ for all $x, y \in X$. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$ the full subcategory of $\mathcal{V}$-Cat whose objects are symmetric. Every $\mathcal{V}$-category carries a natural order defined by $x \leq y$ whenever $k \leq a(x, y)$. A $\mathcal{V}$-category is called *separated* if its natural order is anti-symmetric. We denote by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ the full subcategory of $\mathcal{V}$-Cat whose objects are separated.

Example 3.7 For $\mathcal{V}$ the Boolean quantale, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$ is the category Pos of partially ordered sets and monotone maps, and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category Set of sets and functions. For $\mathcal{V}$ the metric quantale, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ is the category Met of metric spaces and non-expansive maps. For more examples see [13].

We will take advantage of the following useful facts about $\mathcal{V}$-categories. The inclusion functor $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}} \hookrightarrow \mathcal{V}$-$\mathsf{Cat}$ has a left adjoint [27]. It is constructed first by defining the equivalence relation $x \sim y$ whenever $x \leq y$ and $y \leq x$ (where $\leq$ is the natural order introduced earlier). Then this relation induces the separated $\mathcal{V}$-category $(X/_\sim, \tilde{a})$ where $\tilde{a}$ is defined as $\tilde{a}([x],[y]) = a(x, y)$ for every $[x], [y] \in X/_\sim$. Finally the left adjoint of the inclusion functor $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}} \hookrightarrow \mathcal{V}$-$\mathsf{Cat}$ sends every $\mathcal{V}$-category (X, a) to $(X/_\sim, \tilde{a})$. The category $\mathcal{V}$-Cat is autonomous with the tensor $(X, a) \otimes (Y, b) := (X \times Y, a \otimes b)$ where $a \otimes b$ is defined as $(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y')$ and the set of $\mathcal{V}$-functors $\mathcal{V}$-$\mathsf{Cat}((X, a), (Y, b))$ equipped with the map,

$$(f, g) \mapsto \bigwedge_{x \in X} b(f(x), g(x))$$

$\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$, and $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$ inherit the autonomous structure of $\mathcal{V}$-Cat whenever $\mathcal{V}$ is integral [13].

Definition 3.8 A $\mathcal{V}$-Cat-enriched autonomous category C is an autonomous and $\mathcal{V}$-Cat-enriched category C such that the bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is a $\mathcal{V}$-Cat-functor and the adjunction $(- \otimes X) \dashv (X \multimap -)$ is a $\mathcal{V}$-Cat-adjunction. We obtain analogous notions of enriched autonomous category by replacing $\mathcal{V}$-Cat (as basis of enrichment) with $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$, $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym}}$, or $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$.

Example 3.9 The categories Pos, Met, and Set are instances of Definition 3.8.

We now turn our attention to the graded case, more specifically on how to suitably enrich the underlying graded exponential comonad. An obvious way of doing so would be to state that for every $r \in R$ the functor $D_r : \mathsf{C} \to \mathsf{C}$ is $\mathcal{V}$-Cat-enriched. This however turns out to be too strict to soundly interpret the $\mathcal{V}$-compatibility rule concerning promotion (Figure 4). Instead we adopt a more relaxed variant which formally resembles the well-known notion of Lipschitz-continuity from calculus.

Definition 3.10 An $\mathcal{R}$-Lipschitz exponential comonad (for a scalar multiplication $\bullet : R \times \mathcal{V} \to \mathcal{V}$) is an $\mathcal{R}$-graded exponential comonad such that the inequality,

$$r \bullet a(f, g) \leq a(D_r f, D_r g)$$

holds for all C-morphisms $f, g : X \to Y$ and $r \in R$.

Definition 3.11 [Models of graded $\mathcal{V}\lambda$-theories] Consider a graded $\mathcal{V}\lambda$-theory $((G, \Sigma), Ax)$ and a $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$-autonomous category C equipped with an $\mathcal{R}$-Lipschitz exponential comonad. Suppose that for each $X \in G$ we have an interpretation $[\![X]\!]$ as a C-object and analogously for the operation symbols. This interpretation structure is a *model* of the theory if all axioms are satisfied by the interpretation, *i.e.* if $v =_q w$ is an axiom of the theory then $a([\![v]\!], [\![w]\!]) \geq q$.

In the case of *symmetric* graded $\mathcal{V}\lambda$-theories the corresponding notion of a model is obtained by replacing the basis of enrichment (*i.e.* $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sep}}$) by $\mathcal{V}$-$\mathsf{Cat}_{\mathsf{sym,sep}}$.

We can now prove that the $\mathcal{V}$-equational system of graded λ-calculus is sound and complete w.r.t. Definition 3.11.

Theorem 3.12 (Soundness) *Consider a (symmetric) $\mathcal{V}\lambda$-theory $\mathscr{T}$ and a model M of $\mathscr{T}$ over C. If $v =_q w$ is a theorem of $\mathscr{T}$ then $a([\![v]\!], [\![w]\!]) \geq q$.*

Proof. The fact that the equational schema listed in Figure 2 is sound follows from Theorem 2.10 and the definition of a $\mathcal{V}$-category (Definition 3.6). The proof then follows by induction over the rules listed in Figure 4. We only focus on those rules that concern graded modal types (the other ones were already proved in [13]). The case of deriliction follows directly from the fact that for all $X \in |\mathsf{C}|$ the morphism $\epsilon_X : D_1 X \to X$ lives in C and C is $\mathcal{V}$-Cat-enriched. The rules that concern copying and discarding follow from an analogous reasoning. The rule that concerns promotion also follows similarly to the above except that we use the two following properties: first, for all $q, q' \in \mathcal{V}$ and $r \in R$ if $q \geq q'$ then $r \bullet q \geq r \bullet q'$; second, the fact that the graded comonad is Lipschitz. In conjunction both properties entail the implication $a([\![u]\!], [\![u']\!]) \geq q' \Rightarrow a(D_r[\![u]\!], D_r[\![u']\!]) \geq r \bullet q'$. ☐

The completeness result is based on the idea of a *Lindenbaum-Tarski* algebra: it follows from building the syntactic category $\mathrm{Syn}(\mathscr{T})$ of $\mathscr{T}$, showing that it is a model of $\mathscr{T}$, and then showing that if $a([\![v]\!], [\![w]\!]) \geq q$ in $\mathrm{Syn}(\mathscr{T})$ the $\mathcal{V}$-equation $v =_q w$ is a theorem of $\mathscr{T}$. In order to build $\mathrm{Syn}(\mathscr{T})$ and to show that it is indeed a model of $\mathscr{T}$, we resort to the notion of a *multicategory* and associated constructions [34,36,26,38]. More specifically, we will first generate a syntactic multicategory $\mathrm{Syn}_M(\mathscr{T})$ from $\mathscr{T}$ and then show that the former induces an autonomous $\mathrm{Syn}(\mathscr{T})$ with the necessary requisites to be a model of $\mathscr{T}$. The reason we involve multicategories is that some equations we need to face are much more easily proved in this framework, an observation already made in analogous contexts [34,4]. For the same purpose, we also use a bijective correspondence between graded comonads and graded co-Kleisli triples on a multicategory.

Theorem 3.13 (Soundness & Completeness) *For a (symmetric) graded $\mathcal{V}\lambda$-theory $\mathscr{T}$, a $\mathcal{V}$-equation $\Gamma \rhd v =_q w : \mathbb{A}$ is a theorem of $\mathscr{T}$ iff it is satisfied by all models of the theory.*

4 A canonical construction of Lipschitz exponential comonads

This section presents a canonical construction of Lipschitz exponential comonads on $\mathcal{V}$-Cat-autonomous categories that satisfy certain conditions. The construction is inspired by [41], which shows how to build (non-graded) exponential comonads via the notion of a (co)free commutative (co)monoid. In order to describe the connection to *op. cit.* at a suitable level of abstraction, we start with a brief overview of this construction in the form of abstract categorical results. We will then provide a more direct construction.

Let $\mathsf{Mon}_\pi(\mathsf{C})$ be the category of commutative monoids in a symmetric monoidal category C. A crucial observation is that a comonoid in C is the same thing as a monoid in C^{op} [41] – thus the category of commutative comonoids can be seen as $\mathsf{Mon}_\pi(\mathsf{C}^{\mathsf{op}})^{\mathsf{op}}$. The other relevant key observation is that the forgetful functor $\mathsf{Mon}_\pi(\mathsf{C}) \to \mathsf{C}$ is right adjoint if conditions concerning the existence and preservation of a certain limit are met (*cf.* [41]). By duality this induces a forgetful functor $\mathsf{Mon}_\pi(\mathsf{C}^{\mathsf{op}})^{\mathsf{op}} \to \mathsf{C}$ which is furthermore *left* adjoint. Such an adjoint situation induces a comonad on C which can be shown to be exponential (see [41]). Now, we are interested in extending these ideas to the graded setting with $\mathcal{R}$ as the semiring of natural numbers. To this effect we recall next the notion of a strict action.

Definition 4.1 Let M be a monoidal category and C be an arbitrary category. A strict action is a functor $\circledast : \mathsf{M} \times \mathsf{C} \to \mathsf{C}$ that satisfies the following equations for all M-objects m, n and C-objects X:

$$X = I \circledast X \qquad\qquad m \circledast (n \circledast X) = (m \otimes n) \circledast X$$

Consider then both a strict action $\circledast : \mathsf{M} \times \mathsf{A} \to \mathsf{A}$, where M is a discrete category, and an adjoint situation $L \dashv R : \mathsf{A} \to \mathsf{B}$. It is well-known that both constructions yield an M-graded monad on B with $T_n(X) = R(n \circledast LX)$ (see details in [18]). This is the basis to extend [41] to a graded setting.

Specifically let $\mathbb{N}\text{-}\mathsf{Mon}_\pi(\mathsf{C})$ be the category of $(\mathbb{N}, +, 0)$-graded commutative monoids in C. Following an analogous reasoning to the previous paragraphs, one may regard $(\mathbb{N}\text{-}\mathsf{Mon}_\pi(\mathsf{C}^{\mathsf{op}}))^{\mathsf{op}}$ as the category of $(\mathbb{N}, +, 0)$-graded commutative comonoids in C. There is also a forgetful functor $(-)_1 : \mathbb{N}\text{-}\mathsf{Mon}_\pi(\mathsf{C}) \to \mathsf{C}$ which given a graded monoid only keeps the 1-component of the underlying carrier. Then under mild conditions, also pertaining to the existence and preservation of a certain limit $\lim \mathscr{D}$ (details below), this functor is right adjoint. And thus in particular $(-)_1 : (\mathbb{N}\text{-}\mathsf{Mon}_\pi(\mathsf{C}^{\mathsf{op}}))^{\mathsf{op}} \to \mathsf{C}$ is left adjoint. Finally via a few routine calculations one can show the existence of a strict action $\circledast : (\mathbb{N}, \cdot, 1) \times \mathbb{N}\text{-}\mathsf{Mon}_\pi(\mathsf{C}) \to \mathbb{N}\text{-}\mathsf{Mon}_\pi(\mathsf{C})$ defined by,

$$(k, ((X_n)_{n\in\mathbb{N}}, e, f_{m,n} : X_m \otimes X_n \to X_{m+n})) \mapsto ((X_{n\cdot k})_{n\in\mathbb{N}}, e, f_{m\cdot k, n\cdot k} : X_{m\cdot k} \otimes X_{n\cdot k} \to X_{(m+n)\cdot k})$$

Together with the previous adjoint situation this yields an $(\mathbb{N}, \cdot, 1)$-graded comonad on C. By unfolding the respective definitions one can show that this comonad is that of *symmetric powers* described in a very recent publication [37] and stated to be exponential. Due to space constraints we describe only the functorial component. Subsequently we will show that this comonad is Lipschitz under the condition that the aforementioned limit of $\mathscr{D}$ is $\mathcal{V}$-Cat-enriched.

As an instructive first approximation of the $\mathbb{N}$-Lipschitz exponential comonad we intend to describe, consider the map,

$$D : \mathbb{N} \to [\mathsf{C}, \mathsf{C}], n \mapsto \underbrace{\mathrm{Id} \otimes \ldots \otimes \mathrm{Id}}_{n\text{times}}. \tag{3}$$

The assignment D *almost* defines a canonical $(\mathbb{N}, \cdot, 1)$-graded exponential comonad.

Theorem 4.2 *The assignment D of (3) satisfies all the conditions of Definition 2.8 except for symmetry in condition (i) and the last diagram of condition (ii).*

In order to construct an exponential comonad on C one needs to remedy the lack of symmetry of D. To do this, one can consider the sub-$\mathbb{N}$-graded comonad of D which only keeps the *symmetric elements* in the tensor products $D_n X = X^{\otimes n}$. For this we follow the second step of the construction in [41]. Every element σ in the permutation group $\mathrm{Sym}\,(n)$ on n elements defines a natural transformation $D_n \to D_n$ which we also denote by σ. We now define $E : \mathbb{N} \to [\mathsf{C}, \mathsf{C}]$ by mapping $n \in \mathbb{N}$ to the limit E_n of the diagram (4) defined by all these natural transformations. Each E_n is defined on morphisms in the obvious way: if $f : X \to Y$ is a C-morphism then since $D_n f \cdot \sigma = \sigma \cdot D_n f$, the universal property of $E_n Y$ guarantees the existence of a unique C-morphism $E_n f$ that makes Diagram (5) commute.

$$E_n \xrightarrow{\;\epsilon^n\;} D_n \underset{\tau\in\mathrm{Sym}(n)}{\overset{\sigma\in\mathrm{Sym}(n)}{\underset{\cdots}{\rightrightarrows}}} D_n \tag{4}$$

$$\begin{array}{ccccc}
E_nX & \xrightarrow{\;\epsilon^n_X\;} & D_nX & \underset{\tau\in\mathrm{Sym}(n)}{\overset{\sigma\in\mathrm{Sym}(n)}{\underset{\cdots}{\rightrightarrows}}} & D_nX \\
{\scriptstyle E_nf}\Big\downarrow & & {\scriptstyle D_nf}\Big\downarrow & & \Big\downarrow{\scriptstyle D_nf} \\
E_nY & \xrightarrow[\;\epsilon^n_Y\;]{} & D_nY & \underset{\tau\in\mathrm{Sym}(n)}{\overset{\sigma\in\mathrm{Sym}(n)}{\underset{\cdots}{\rightrightarrows}}} & D_nY
\end{array} \tag{5}$$

Theorem 4.3 *Suppose that for every* C*-object* X*,* $(X \otimes -)$ *preserves the limits* (4)*. Then the assignment* E *defined by the limits* (4) *induces a sub-*$\mathbb{N}$*-graded comonad of* D *which is furthermore an* $\mathbb{N}$*-graded exponential comonad.*

We will now show that the graded comonad E is additionally Lipschitz. First we define the following scalar multiplication.

Proposition 4.4 *For any commutative quantale* $\mathcal{V}$*, the map* $\bullet : \mathbb{N} \times \mathcal{V} \to \mathcal{V}$ *defined by,*

$$n \bullet q = \underbrace{q \otimes \ldots \otimes q}_{n \text{ times}} \text{ if } n \neq 0 \qquad 0 \bullet q = k$$

is a scalar multiplication in the sense of Definition 3.4.

Proof. To see that $n \bullet -$ preserves arbitrary joins we compute,

$$\begin{aligned}
& n \bullet \left(\bigvee X\right) \\
& \triangleq \left(\bigvee X\right) \otimes \cdots \otimes \left(\bigvee X\right) \\
& = \bigvee (X \otimes \cdots \otimes X) && \{\otimes \text{ preserves joins}\} \\
& = \bigvee \{x_1 \otimes \cdots \otimes x_n \mid x_1, \ldots, x_n \in X\} \\
& = \bigvee \{x \otimes \cdots \otimes x \mid x \in X\} && \{\star\} \\
& \triangleq \bigvee n \bullet X
\end{aligned}$$

where the step marked with $(\star)$ follows from the fact that the inequation below holds.

$$x_1 \otimes \cdots \otimes x_n \leq \left(\bigvee\{x_1, \ldots, x_n\}\right) \otimes \cdots \otimes \left(\bigvee\{x_1, \ldots, x_n\}\right) \qquad (x_1, \ldots, x_n \in X)$$

□

Next, let C be a $\mathcal{V}$-Cat-autonomous category and the underlying diagram of (4) for a C-object X be denoted by $\mathscr{D}$. Also assume that for every two cones $f, g : A \to X^{\otimes n}$ for $\mathscr{D}$ the equation $a(f, g) = a(f', g')$ holds where $f', g' : A \to E_n(X)$ are the corresponding mediating morphisms. More compactly this amounts to the statement that C has the $\mathcal{V}$-Cat-limit of $\mathscr{D}$ weighted by the functor ! (constant on the $\mathcal{V}$-Cat-object 1). This condition guarantees that E is $\mathbb{N}$-Lipschitz.

Theorem 4.5 *Consider a* $\mathcal{V}$-Cat*-autonomous category* C *such that it has the* $\mathcal{V}$-Cat*-limit of* $\mathscr{D}$ *weighted by* ! *and additionally assume that for every* C*-object* X *the functor* $(X \otimes -)$ *preserves this limit, then* E *is an* $\mathbb{N}$*-Lipschitz exponential comonad.*

Proof. Consider two C-morphisms $f, g : X \to Y$. We reason,

$$
\begin{aligned}
& n \bullet a(f,g) \\
& \triangleq \underbrace{a(f,g) \otimes \ldots \otimes a(f,g)}_{n \text{ times}} \\
& \leq a(f^{\otimes n}, g^{\otimes n}) && \{\otimes \text{ in } \mathsf{C} \text{ is } \mathcal{V}\text{-}\mathsf{Cat}\text{-enriched}\} \\
& \triangleq a(D_n f, D_n g) \\
& \leq a(D_n f \cdot \epsilon^n_X, D_n g \cdot \epsilon^n_X) && \{\mathsf{C} \text{ is } \mathcal{V}\text{-}\mathsf{Cat}\text{-enriched}\} \\
& = a(E_n f, E_n g) && \{\text{limit of } \mathscr{D} \text{ is } \mathcal{V}\text{-}\mathsf{Cat}\text{-enriched}\}
\end{aligned}
$$

□

5 Applications to timed and probabilistic computation

5.1 Timed computation and dilations

We now revisit the example of wait calls from §1 and equip it with a concrete model by applying the canonical construction of $\mathbb{N}$-Lipschitz exponential comonads detailed in §4. Recall that the example is based on a ground type X and a signature $\{\mathtt{wait_n} : X \to X \mid n \in \mathbb{N}\}$ of wait calls. Consider then the following metric axioms proposed in [13]:

$$\mathtt{wait_0}(x) =_0 x \qquad \mathtt{wait_n}(\mathtt{wait_m}(x)) =_0 \mathtt{wait_{n+m}}(x) \qquad \frac{\epsilon = |m - n|}{\mathtt{wait_n}(x) =_\epsilon \mathtt{wait_m}(x)} \tag{6}$$

In order to apply the construction in §4, we need first of all a Met-enriched autonomous category. For this case we choose Met itself (*cf.* Example 3.9). Next we show that the tensor $\otimes$ in Met preserves all limits; actually we prove the following more general claim.

Proposition 5.1 *Let $\mathcal{V}$ be a quantale whose operation $\otimes$ preserves arbitrary meets and let us consider the respective category $\mathcal{V}$-Cat. For every $\mathcal{V}$-category X the functor $(- \otimes X) : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ preserves all limits. The same property holds for the cases $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sep}}$, $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym}}$, and $\mathcal{V}\text{-}\mathsf{Cat}_{\mathsf{sym,sep}}$.*

Corollary 5.2 *For all categories C mentioned in Example 3.9 (which includes Met) and C-objects X the functor $(- \otimes X) : \mathsf{C} \to \mathsf{C}$ preserves all limits.*

Finally, it is straightforward to prove that Met has the $\mathcal{V}$-Cat-limit of $\mathscr{D}$ weighted by ! and therefore all pre-requisites of the construction are satisfied. By unfolding the respective definitions we deduce that $E_n(X)$ is the metric space whose elements are n-copies $(x, \ldots, x)$ of an element $x \in X$ and whose metric is the restriction of the metric in $X^{\otimes n}$. The counit is the identity and comultiplication amounts to rebracketing. The operation $d^{m,n}$ amounts to rebracketing as well. It is then easy to build a model for the metric theory of wait calls that was previously presented: fix a metric space A, interpret the ground type X as $\mathbb{N} \otimes A$ and the operation symbol $\mathtt{wait_n} : X \to X$ as the non-expansive map $[\![\mathtt{wait_n}]\!] : \mathbb{N} \otimes A \to \mathbb{N} \otimes A, (i, a) \mapsto (i + n, a)$. It only remains to prove that the axioms in (6) are satisfied by the proposed interpretation, but this can be shown via a few routine calculations.

We end this subsection by relating the comonad that we canonically obtained to the comonad of dilations presented in [30]. The latter's main idea is that of distance dilation: given a metric space (X, d) we obtain a new one $\mathrm{Dil}_n(X, d) := (X, n \bullet d)$ by scaling up distances via multiplication, more concretely $(n \bullet d)(x, y) = n \bullet d(x, y)$ for $n \in \mathbb{N}$ and $x, y \in X$. It is easy to see that $E_n \cong \mathrm{Dil}_n$ and moreover that the underlying comonadic operations agree. It is also easy to see that the copy, discard, and monoidal operations agree as well. This yields the following result.

Corollary 5.3 *The Met-autonomous category Met of metric spaces and non-expansive maps equipped with the comonad of dilations yields a model of the metric theory of wait calls (6).*

5.2 Probabilistic computation

[13, Example 28] presents a metric equational system to reason about the total variation distance between distributions constructed as probabilistic programs, specifically individual steps in non-standard random walks. It is however cumbersome to reason about distances between random walks consisting of n steps when they are expressed in a purely linear language. This is because a probabilistic term like $\texttt{normal}(0,1)$ operationally corresponds to a *single* sample which cannot be copied. Thus, to write a program using n normal deviates we need to call n i.i.d. samples from $\underbrace{\texttt{normal}(0,1) \otimes \ldots \otimes \texttt{normal}(0,1)}_{n \text{ times}}$ which is inconvenient and unclear (especially for large values of n), but also difficult to maintain and generalise. Using a graded system, we can not only assume a clean and parametric access to such i.i.d. samples but also to more complex sampling schemes (details below). Furthermore, we have a convenient way of manipulating such sequences of samples via the promotion rule, and to feed them into n-ary functions through the copy (*i.e.* contraction) rule. All of this whilst maintaining the ability to reason about distances between programs.

Let us illustrate our previous remarks with some simple examples. We start by briefly presenting a toy probabilistic language (more details can be found in [13]). We consider only two ground types $\texttt{real}$ and $\texttt{real}^+$ (in particular, we will view the integers 0 and 1 as reals). The graded modal type $!_n\, \texttt{real}$ can then be thought of as the type of n real samples. We also consider a signature of operations consisting of the real numbers $\{r : \mathbb{I} \to \texttt{real} \mid r \in \mathbb{Q}\}$, the addition and multiplication operations $+, * : \texttt{real}, \texttt{real} \to \texttt{real}$, and finally three collections of built-in samplers which we detail next. The first collection consists of samplers returning k samples from an urn containing m balls labelled 0 and n balls labelled 1 *with replacement* (*i.e.* we return the ball to the urn after reading its value). We denote the samplers of this class $\texttt{replace}(k,m,n) : !_k\, \texttt{real}$. The second collection samples from the same urn model but *without replacement*. We denote these samplers $\texttt{no\_replace}(k,m,n) : !_k\, \texttt{real}$ (and of course require that $k \leq m + n$). The third class $\texttt{iid\_normal}(k;\mu,\sigma) : \texttt{real}, \texttt{real}^+ \to !_k\, \texttt{real}$ will simply sample k i.i.d. normal deviates.

We proceed by providing a concrete graded λ-model for the language. First we fix the category Ban of Banach spaces and linear contractions as our Met-enriched autonomous category (see [31,12,15,13] for more details about this style of semantics). Specifically Ban is autonomous when equipped with the projective tensor product $\hat{\otimes}_\pi$ and the internal hom $\multimap$ defined as the space of *bounded* linear maps equipped with the sup-norm [49]. It is also straightforward to prove that Ban has the Met-limit of $\mathscr{D}$ weighted by !. Then in order to apply the construction in §4 we use the following result.

Proposition 5.4 *For every Banach space W and every $n \in \mathbb{N}$, the functor $(-\otimes W) : \mathsf{Ban} \to \mathsf{Ban}$ preserves the limit of diagram (4) which defines E_n in terms of all the permutations $\sigma \in \mathrm{Sym}\,(n)$.*

Proof. The proof is inspired by an analogous one in [12] and hinges on the fact that the contraction $\epsilon^n : E_n(V) \to V^{\otimes n}$ is split mono. To prove the latter, let us consider the symmetrisation operator $\frac{1}{n!}\sum_{\sigma \in \mathrm{Sym}(n)} \sigma : V^{\otimes n} \to V^{\otimes n}$ [6,8] – it is a contraction because the properties of norms entail,

$$\left\| \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \sigma \right\| \leq \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \|\sigma\| = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} 1 = 1$$

It is then straightforward to show that this operator restricts on the codomain to a linear map $\partial^n : V^{\otimes n} \to E_n(V)$ by taking advantage of the fact that $\mathrm{Sym}\,(n)$ is a group. Moreover $E_n(V)$ inherits its norm from $V^{\otimes n}$ which yields $\|\partial^n\| = \|\frac{1}{n!}\sum_{\sigma \in \mathrm{Sym}(n)} \sigma\|$. Thus ∂^n is a linear contraction as well. Next, in order to prove that ∂^n is a retraction of ϵ^n consider the following facts. By construction we have $\sigma \cdot \epsilon^n = \epsilon^n$ for all symmetries $\sigma \in \mathrm{Sym}\,(n)$ which gives rise to the equation $\left(\frac{1}{n!}\sum_{\sigma \in \mathrm{Sym}(n)} \sigma\right) \cdot \epsilon^n = \partial^n \cdot \epsilon^n = \mathsf{id} \cdot \epsilon^n$. Moreover ϵ^n is an inclusion. Therefore for every vector $v \in E_n(V)$ we obtain,

$$\partial^n(\epsilon^n(v)) = \tfrac{1}{n!}\textstyle\sum_{\sigma \in \mathrm{Sym}(n)} \sigma(\epsilon^n(v)) = \epsilon^n(v) = v$$

The final step is to prove that every cone $f : U \to V^{\otimes n} \hat{\otimes}_\pi W$ factorises uniquely through $\epsilon^n \otimes \mathsf{id}$. By composition we obtain a linear contraction $(\partial^n \otimes \mathsf{id}) \cdot f : U \to E_n(V) \hat{\otimes}_\pi W$. Let us show that it factorises f through $\epsilon^n \otimes \mathsf{id}$. Consider a vector $u \in U$. By construction we know that $f(u) = \sigma \otimes \mathsf{id}\,(f(u))$ for all permutations σ on n. This entails,

$$
\begin{aligned}
& f(u) \\
&= \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \sigma \otimes \mathsf{id}\,(f(u)) \\
&= \frac{1}{n!} \left(\sum_{\sigma \in \mathrm{Sym}(n)} \sigma \right) \otimes \mathsf{id}\,(f(u)) && \{\text{Addition distributes over } (- \otimes \mathsf{id})\} \\
&= \left(\frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}(n)} \sigma \right) \otimes \mathsf{id}\,(f(u)) && \{\text{Scaling distributes over } (- \otimes \mathsf{id})\} \\
&= \partial^n \otimes \mathsf{id}\,(f(u))
\end{aligned}
$$

We thus obtain the chain of equalities $(\epsilon^n \otimes \mathsf{id}) \cdot (\partial^n \otimes \mathsf{id})(f(u)) = (\epsilon^n \otimes \mathsf{id})(f(u)) = f(u)$. Finally unicity follows from the fact that ϵ^n is split mono. ☐

This yields a canonical $\mathbb{N}$-Lipschitz exponential comonad on Ban, and we can interpret $[\![!_n\, \texttt{real}]\!] \triangleq E_n[\![\texttt{real}]\!] = E_n(\mathcal{M}\mathbb{R})$ where $\mathcal{M}\mathbb{R}$ is the Banach space of finite measures on $\mathbb{R}$. Note that the elements of $[\![!_n\, \texttt{real}]\!]$ are invariant under all permutations in $\mathrm{Sym}\,(n)$, but need not in general be i.i.d. distributions. For example $[\![\texttt{replace}(k,m,n)]\!]$ corresponds to the i.i.d. case as it is given by the k-fold tensor of the distribution $\mathrm{Bern}(n/(n+m))$, but $[\![\texttt{no\_replace}(k,m,n)]\!]$ is permutation-invariant without being i.i.d. Quite a lot is know about permutation-invariant distributions like these, usually known as *finite exchangeable sequences* in the probabilistic literature. In particular, [17] shows that the following metric axiom is sound.

$$\overline{\texttt{replace}(k,m,n) =_{4k/(m+n)} \texttt{no\_replace}(k,m,n)} \tag{7}$$

The denotation of $\texttt{iid\_normal}(k;\mu,\sigma)$ is the linear, norm-1 operator defined by the Markov kernel $\mathbb{R} \times \mathbb{R}^+ \to (\mathcal{M}\mathbb{R})^{\otimes n} \to \mathcal{M}(\mathbb{R}^n), (\mu,\sigma) \mapsto \mathrm{Normal}(\mu,\sigma)^{\otimes n}$. There is no known closed-form expression for the total variation distance between Gaussian distributions. However, upper bounds are known. In particular, following [16, Prop. 1.2], we know that the metric axiom below is sound.

$$\overline{\texttt{iid\_normal}(k;\mu_1,\sigma_1) =_{\phi(\mu_1,\sigma_1,\mu_2,\sigma_2)} \texttt{iid\_normal}(k;\mu_2,\sigma_2)} \tag{8}$$

where $\phi(\mu_1,\sigma_1,\mu_2,\sigma_2) = \frac{1}{2}\sqrt{k\left(\frac{\sigma_2^2 - \sigma_1^2 + (\mu_1-\mu_2)^2}{\sigma_1^2} - \log\left(\frac{\sigma_2^2}{\sigma_1^2}\right)\right)}$.

Now based on these axioms, and the metric equational rules of Fig. 4 we can easily bound the total variation distance between the final position of two complex k-steps random walks of the type used in Monte-Carlo simulations (*e.g.* to value options [28]). For example, consider first the random walk on $\mathbb{R}$ where at each step the *sign* of the jump is determined by a sample from $\texttt{replace}(k,m,n)$ and its *magnitude* by a sample from $\texttt{iid\_normal}(k;\mu_1,\sigma_1)$. Suppose we want to bound the distance of this walk with one whose sign is sampled from $\texttt{no\_replace}(k,m,n)$ and magnitude from $\texttt{iid\_normal}(k;\mu_2,\sigma_2)$ instead. Working directly at the level of the semantics, this would be a highly non-trivial task, however if we express these walks as programs in our graded system we can straightforwardly compute such a bound.

The walks can be programmed as follows:

$$\texttt{walk1} \triangleq \texttt{pr}_{k,[1,1]}\ \texttt{replace}(k,m,n), \texttt{iid\_normal}(k;\mu_1,\sigma_1)\ \texttt{fr}\ x,y.\ (2*\texttt{dr}(x)-1)*\texttt{dr}(y) :\ !_k\texttt{real}$$
$$\texttt{walk2} \triangleq \texttt{pr}_{k,[1,1]}\ \texttt{no\_replace}(k,m,n), \texttt{iid\_normal}(k;\mu_2,\sigma_2)\ \texttt{fr}\ x,y.\ (2*\texttt{dr}(x)-1)*\texttt{dr}(y) :\ !_k\texttt{real}$$
$$\texttt{endpoint}(w) \triangleq \texttt{cp}_{(1,\ldots,1)}\ w\ \texttt{to}\ x_1,\ldots,x_n.\ \texttt{dr}(x_1)+\ldots+\texttt{dr}(x_n) : \texttt{real}.$$

Using the metric axioms (7)-(8) and Fig. 4, the bound can be straightforwardly checked to be $\texttt{endpoint(walk1)} =_{4k/(m+n)+\phi(\mu_1,\sigma_1,\mu_2,\sigma_2)} \texttt{endpoint(walk2)}$. The higher-order features of the language would allow us to write the program above more modularly by introducing an iterator and still reason quantitatively about it. We chose the shorter, less modular presentation above in the interest of brevity.

6 Conclusions and future work

We presented a sound and complete $\mathcal{V}$-equational system for a graded λ-calculus via the notion of a Lipschitz exponential comonad. We showed how to build such comonads canonically via a universal construction and applied our results to both timed and probabilistic computation. There are multiple research lines which we intend to explore next. First, we believe that the construction of Lipschitz exponential comonads is interesting *per se* and that it deserves further exploration from a more categorical perspective. For example, we are interested in knowing whether the adjunction involved is monoidal and whether it arises from the development of general results about graded (co)equational theories over (enriched) monoidal categories. Second, our results were applied to the setting of metric equations only but they go beyond that – in particular, we would like to explore as well the inequational, ultra-metric, and fuzzy cases due to their increasing relevance in the literature. Third, whilst we presented relatively straightforward metric equational theories and corresponding models for timed and probabilistic computation, we are also interested in knowing whether the same can be done for hybrid [42,24] and quantum [43,14] computation, two rapidly emerging paradigms with an intrinsically quantitative nature. Finally we are also interested in knowing if there is any formal connection with previous work on the notion of comonadic lax extension and the relational semantics involving modal types [32,1]

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