Partial Frames, Their Free Frames
and Their Congruence Frames

Anneliese Schauerte[1]   John Frith[2]

[a] *Department of Mathematics and Applied Mathematics*
University of Cape Town
Cape Town, South Africa

Abstract

The context of this work is that of partial frames; these are meet-semilattices where not all subsets need have joins. A selection function, $\mathcal{S}$, specifies, for all meet-semilattices, certain subsets under consideration, which we call the "designated" ones; an $\mathcal{S}$-frame then must have joins of (at least) all such subsets and binary meet must distribute over these. A small collection of axioms suffices to specify our selection functions; these axioms are sufficiently general to include as examples of partial frames, bounded distributive lattices, σ-frames, κ-frames and frames.
We consider right and left adjoints of $\mathcal{S}$-frame maps, as a prelude to the introduction of closed and open maps.
Then we look at what might be an appropriate notion of Booleanness for partial frames. The obvious candidate is the condition that every element be complemented; this concept is indeed of interest, but we pose three further conditions which, in the frame setting, are all equivalent to it. However, in the context of partial frames, the four conditions are distinct. In investigating these, we make essential use of the free frame over a partial frame and the congruence frame of a partial frame. We compare congruences of a partial frame, technically called $\mathcal{S}$-congruences, with the frame congruences of its free frame. We provide a natural transformation for the situation and also consider right adjoints of the frame maps in question. We characterize the case where the two congruence frames are isomorphic and provide examples which illuminate the possible different behaviour of the two.
We conclude with a characterization of closedness and openness for the embedding of a partial frame into its free fame, and into its congruence frame.

Keywords: frame, partial frame, $\mathcal{S}$-frame, κ-frame, σ-frame, free frame over partial frame, congruence frame, Boolean algebra, closed map, open map

1 Introduction

Partial frames are meet-semilattices where, in contrast with frames, not all subsets need have joins. A selection function, $\mathcal{S}$, specifies, for all meet-semilattices, certain subsets under consideration, which we call the "designated" ones; an $\mathcal{S}$-frame then must have joins of (at least) all such subsets and binary meet must distribute over these. A small collection of axioms suffices to specify our selection functions; these axioms are sufficiently general to include as examples of partial frames, bounded distributive lattices, σ-frames, κ-frames and frames.

[1] Email: Anneliese.Schauerte@uct.ac.za
[2] Email: John.Frith@uct.ac.za

We consider the classical notions of right and left adjoints, for $\mathcal{S}$-frame maps. Unlike the situation for full frames, such maps need not have right adjoints. This is a prelude to the introduction of closed and open maps, and a discussion of their properties.

What is an appropriate notion of Booleanness for partial frames? The obvious answer is that the partial frame should have every element complemented; this concept is indeed of interest, but we pose three further conditions which, in the frame setting, are all equivalent to it. However, in the context of partial frames, the four conditions are distinct. In investigating these, we make essential use of the free frame over a partial frame and the congruence frame of a partial frame.

We compare congruences of a partial frame, technically called $\mathcal{S}$-congruences, with the frame congruences of its free frame. We provide a natural transformation for the situation and also consider right adjoints of the frame maps in question. We characterize the case where the two congruence frames are isomorphic and provide examples which illuminate the possible different behaviour of the two.

We conclude with a characterization of closedness and openness for the embedding of a partial frame into its free fame, and into its congruence frame.

Since this document is intended as an extended abstract, proofs are omitted.

2   Background

This background section is taken largely from [16]. See [22] and [17] as references for frame theory; see [3] and [2] for σ-frames; see [19] and [20] for κ-frames; see [18] and [1] for general category theory.

The basics of our approach to partial frames can be found in [4], [5] and [7]. Our papers with a more topological flavour are [6], [8], [10], [13] and [14]. Our papers with a more algebraic flavour are [9], [11] and [12]. Crucial for this paper is [15]. We are indebted to earlier work by other authors in this field: see [21], [24], [25] and [23]. For those interested in a comparison of the various approaches, see [5].

A *meet-semilattice* is a partially ordered set in which all finite subsets have a meet. In particular, we regard the empty set as finite, so a meet-semilattice comes equipped with a top element, which we denote by 1. We do not insist that a meet-semilattice should have a bottom element, which, if it exists, we denote by 0. A function between meet-semilattices $f : L \to M$ is a *meet-semilattice map* if it preserves finite meets, as well as the top element. A *sub meet-semilattice* is a subset for which the inclusion map is a meet-semilattice map.

The essential idea for a *partial frame* is that it should be "frame-like" but that not all joins need exist; only certain joins have guaranteed existence and binary meets should distribute over these joins. The guaranteed joins are specified in a global way on the category of meet-semilattices by specifying what is called a selection function; the details are given below.

Definition 2.1 A *selection function* is a rule, which we usually denote by $\mathcal{S}$, which assigns to each meet-semilattice A a collection $\mathcal{S}A$ of subsets of A, such that the following conditions hold (for all meet-semilattices A and B):

   (S1) For all $x \in A$, $\{x\} \in \mathcal{S}A$.
   (S2) If $G, H \in \mathcal{S}A$ then $\{x \wedge y : x \in G, y \in H\} \in \mathcal{S}A$.
   (S2)$'$ If $G, H \in \mathcal{S}A$ then $\{x \vee y : x \in G, y \in H\} \in \mathcal{S}A$.
   (S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then

$$\bigcup_{x \in G} H_x \in \mathcal{S}A.$$

   (S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}(f[A]) = \{f[G] : G \in \mathcal{S}A\} \subseteq \mathcal{S}B.$$

 (SSub) For any sub meet-semilattice B of meet-semilattice A, if $G \subseteq B$ and $G \in \mathcal{S}A$, then $G \in \mathcal{S}B$.

(SFin) If F is a finite subset of A, then $F \in \mathcal{S}A$.
(SCov) If $G \subseteq H$ and $H \in \mathcal{S}A$ with $\bigvee H = 1$ then $G \in \mathcal{S}A$. (Such H are called *$\mathcal{S}$-covers*.)
(SRef) Let $X, Y \subseteq A$. If $X \leq Y$ with $X \in \mathcal{S}A$ there is a $C \in \mathcal{S}A$ such that $X \leq C \subseteq Y$. (By $X \leq Y$ we mean, as usual, that for each $x \in X$ there exists $y \in Y$ such that $x \leq y$.)

Of course (SFin) implies (S1) but there are situations where we do not impose (SFin) but insist on (S1). Note that we always have $\emptyset \in \mathcal{S}A$. Once a selection function, $\mathcal{S}$, has been fixed, we speak informally of the members of $\mathcal{S}A$ as the *designated* subsets of A.

Definition 2.2 An *$\mathcal{S}$-frame* L is a meet-semilattice in which every designated subset has a join and for any such designated subset B of L and any $a \in L$,

$$a \wedge \bigvee B = \bigvee_{b \in B} a \wedge b.$$

Of course such an $\mathcal{S}$-frame has both a top and a bottom element which we denote by 1 and 0 respectively. A meet-semilattice map $f : L \to M$, where L and M are $\mathcal{S}$-frames, is an *$\mathcal{S}$-frame map* if $f(\bigvee B) = \bigvee_{b \in B} f(b)$ for any designated subset B of L. In particular such an f preserves the top and bottom element.
A *sub $\mathcal{S}$-frame* T of an $\mathcal{S}$-frame L is a subset of L such that the inclusion map $i : T \to L$ is an $\mathcal{S}$-frame map.
The category $\mathcal{S}\mathbf{Frm}$ has objects $\mathcal{S}$-frames and arrows $\mathcal{S}$-frame maps.

We use the terms "partial frame" and "$\mathcal{S}$-frame" interchangeably, especially if no confusion about the selection function is likely. We also use the term *full frame* in situations where we wish to emphasize that all joins exist.

Note 1 *Here are some examples of different selection functions and their corresponding $\mathcal{S}$-frames.*

1. *In the case that all joins are specified, we are of course considering the notion of a frame.*

2. *In the case that (at most) countable joins are specified, we have the notion of a σ-frame.*

3. *In the case that joins of subsets with cardinality less than some (regular) cardinal κ are specified, we have the notion of a κ-frame.*

4. *In the case that only finite joins are specified, we have the notion of a bounded distributive lattice.*

The remainder of this section gives a lot of information about $\mathcal{H}_{\mathcal{S}}L$, the free frame over the $\mathcal{S}$-frame L, as well as $\mathcal{C}_{\mathcal{S}}L$, the frame of $\mathcal{S}$-congruences of L, and the relationship between the two. These results come from [7] on $\mathcal{H}_{\mathcal{S}}L$, [9] and [11] on $\mathcal{C}_{\mathcal{S}}L$.
In the definition below, L is an $\mathcal{S}$-frame.

Definition 2.3 (a) A subset J of an L is an *$\mathcal{S}$-ideal of L* if J is a non-empty downset closed under designated joins (the latter meaning that if $X \subseteq J$, for X a designated subset of L, then $\bigvee X \in J$).
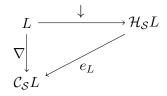
(b) The collection of all $\mathcal{S}$-ideals of L will be denoted $\mathcal{H}_{\mathcal{S}}L$, and called the *$\mathcal{S}$-ideal frame of L*. It is in fact the free frame over L.

(c) For $I \in \mathcal{H}_{\mathcal{S}}L$, $t \in (\downarrow x) \vee I \Longleftrightarrow t \leq x \vee s$, for some $s \in I$.

(d) We call $\theta \subseteq L \times L$ an *$\mathcal{S}$-congruence on L* if it satisfies the following:
   (C1) θ is an equivalence relation on L.
   (C2) $(a, b), (c, d) \in \theta$ implies that $(a \wedge c, b \wedge d) \in \theta$.
   (C3) If $\{(a_\alpha, b_\alpha) : \alpha \in \mathcal{A}\} \subseteq \theta$ and $\{a_\alpha : \alpha \in \mathcal{A}\}$ and $\{b_\alpha : \alpha \in \mathcal{A}\}$ are designated subsets of L, then $(\bigvee_{\alpha \in \mathcal{A}} a_\alpha, \bigvee_{\alpha \in \mathcal{A}} b_\alpha) \in \theta$.

(e) The collection of all $\mathcal{S}$-congruences on L is denoted by $\mathcal{C}_{\mathcal{S}}L$; it is in fact a (full) frame with meet given by intersection.

(f) (i) Let $A \subseteq L \times L$. We use the notation $\langle A \rangle$ to denote the smallest $\mathcal{S}$-congruence containing A. This exists by completeness of $\mathcal{C}_\mathcal{S}L$.

    (ii) We define $\nabla_a = \{(x,y) : x \vee a = y \vee a\}$ and $\Delta_a = \{(x,y) : x \wedge a = y \wedge a\}$; these are $\mathcal{S}$-congruences on L.

    (iii) It is easily seen that $\nabla_a = \bigcap\{\theta : \theta \in \mathcal{C}_\mathcal{S}L$ and $(0,a) \in \theta\} = \langle(0,a)\rangle$ and that $\Delta_a = \bigcap\{\theta : \theta \in \mathcal{C}_\mathcal{S}L$ and $(a,1) \in \theta\} = \langle(a,1)\rangle$.

    (iv) For $a \leq b$, it follows that $\Delta_a \cap \nabla_b = \langle(a,b)\rangle$.

    (v) The congruence $\nabla_1 = L \times L$ is the top element and $\nabla_0 = \{(x,x) : x \in L\}$ (called the *diagonal*) is the bottom element of $\mathcal{C}_\mathcal{S}L$.

(g) The following hold in $\mathcal{C}_\mathcal{S}L$.

    (i) For any $\theta \in \mathcal{C}_\mathcal{S}L$, $\theta = \bigvee\{\nabla_b \wedge \Delta_a : (a,b) \in \theta, a \leq b\}$.

    (ii) $\nabla_a \vee \theta = \{(x,y) : (x \vee a, y \vee a) \in \theta\}$.

    (iii) $\Delta_a \vee \theta = \{(x,y) : (x \wedge a, y \wedge a) \in \theta\}$.

    (iv) For any $I \in \mathcal{H}_\mathcal{S}L$, $\bigvee_{x \in I} \nabla_x = \bigcup_{x \in I} \nabla_x$.

(h) The function $\nabla : L \to \mathcal{C}_\mathcal{S}L$ given by $\nabla(a) = \nabla_a$ is an $\mathcal{S}$-frame embedding. It has the universal property that if $f : L \to M$ is an $\mathcal{S}$-frame map into a frame M with complemented image, then there exists a frame map $\bar{f} : \mathcal{C}_\mathcal{S}L \to M$ such that $f = \bar{f} \circ \nabla$.

(i) We also note that for frame maps f and g with domain $\mathcal{C}_\mathcal{S}L$, if $f \circ \nabla = g \circ \nabla$ then $f = g$.

(j) A useful congruence for our purposes is the *Madden* congruence, denoted π_L below:

    (i) For $x \in L$, set $P_x = \{t \in L : t \wedge x = 0\}$.

    (ii) For $x \in L$, P_x is an $\mathcal{S}$-ideal, and in $\mathcal{H}_\mathcal{S}L$, $P_x = (\downarrow x)^*$, the pseudocomplement of $\downarrow x$.

    (iii) Let $\pi_L = \{(x,y) : P_x = P_y\}$; π_L is an $\mathcal{S}$-congruence.

    (iv) The quotient map induced by the Madden congruence, $p : L \to L/\pi_L$ is dense, onto and the universal such. We refer to this as the *Madden quotient* of L. (See [11].)

Definition 2.4 For any $\mathcal{S}$-frame L, define $e_L : \mathcal{H}_\mathcal{S}L \to \mathcal{C}_\mathcal{S}L$ to be the unique frame map such that $e_L(\downarrow a) = \nabla_a$ for all $a \in L$; that is, making the following diagram commute:

$$
\begin{array}{ccc}
L & \xrightarrow{\;\;\downarrow\;\;} & \mathcal{H}_\mathcal{S}L \\
\nabla \downarrow & \swarrow {\scriptstyle e_L} & \\
\mathcal{C}_\mathcal{S}L & &
\end{array}
$$

That this map e_L exists follows from the freeness of $\mathcal{H}_\mathcal{S}L$ as a frame over L. See [7].

Note 2 *For any $\mathcal{S}$-frame L, $\mathcal{H}_\mathcal{S}L$ is isomorphic to a subframe of $\mathcal{C}_\mathcal{S}L$; that is, the free frame over L is isomorphic to a subframe of the frame of $\mathcal{S}$-congruences of L.*

3   Right and left adjoints

We use the following standard terminology:

Definition 3.1 Let $h : L \to M$ be an $\mathcal{S}$-frame map.
A function $r : M \to L$ is a *right adjoint* of h if

$$h(x) \leq m \iff x \leq r(m) \text{ for all } x \in L, m \in M.$$

A function $l : M \to L$ is a *left adjoint* of h if

$$l(m) \leq x \iff m \leq h(x) \text{ for all } x \in L, m \in M.$$

We make no claim that all $\mathcal{S}$-frame maps have right (or left) adjoints; this is false (see Example 3.3). However, clearly if an $\mathcal{S}$-frame map has a right or left adjoint, such is unique.

Lemma 3.2 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(i) *If h has a right adjoint r, then for all $m \in M$,*

$$r(m) = \bigvee\{x \in L : h(x) \leq m\}.$$

(ii) *If h has a left adjoint ℓ, then for all $m \in M$,*

$$l(m) = \bigwedge\{x \in L : m \leq h(x)\}.$$

We note that the existence of the above joins and meets has to be established since an $\mathcal{S}$-frame need not be complete.

Example 3.3 This is an example of an $\mathcal{S}$-frame map which has neither a right nor a left adjoint.

Let L be the σ-frame consisting of all countable and cocountable subsets of $\mathbb{R}$, and $\mathbf{2}$ denote the 2-element chain. Define $h : L \to \mathbf{2}$ by $h(C) = 0$ if C is countable and $h(D) = 1$ if D is cocountable. Then h is a σ-frame map. However it has no right adjoint since there is no largest $A \in L$ with $h(A) = 0$. Similarly it has no left adjoint.
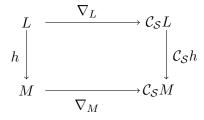
Proposition 3.4 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(i) *Suppose that h has a right adjoint, r. Then h preserves all existing joins and r preserves all existing meets.*

(ii) *Suppose that h has a left adjoint L. Then h preserves all existing meets and ℓ preserves all existing joins.*

4   Closed and open maps

Definition 4.1 Let $h : L \to M$ be an $\mathcal{S}$-frame map.

We call h *closed* if, for all $m \in M$, there exists $x \in L$ with $(h \times h)^{-1}(\nabla_m) = \nabla_x$.

We call h *open* if, for all $m \in M$, there exists $x \in L$ with $(h \times h)^{-1}(\Delta_m) = \Delta_x$.

We know that (see [11]) that $\mathcal{C}_{\mathcal{S}}$ is a functor from $\mathcal{S}$-frames to frames which is natural in the sense that for any $\mathcal{S}$-frame $h : L \to M$ we have a frame map $\mathcal{C}_{\mathcal{S}} h : \mathcal{C}_{\mathcal{S}} L \to \mathcal{C}_{\mathcal{S}} M$ making the following diagram commute:

$$
\begin{array}{ccc}
L & \xrightarrow{\ \nabla_L\ } & \mathcal{C}_{\mathcal{S}} L \\
\scriptstyle h \downarrow & & \downarrow \scriptstyle \mathcal{C}_{\mathcal{S}} h \\
M & \xrightarrow[\ \nabla_M\ ]{} & \mathcal{C}_{\mathcal{S}} M
\end{array}
$$

Now $(h \times h)^{-1}$ is the right adjoint of $\mathcal{C}_{\mathcal{S}} h$, because, for $\theta \in \mathcal{C}_{\mathcal{S}} L$, $\mathcal{C}_{\mathcal{S}} h(\theta)$ is the $\mathcal{S}$-congruence of M generated by $(h \times h)[\theta]$, so for all $\theta \in \mathcal{C}_{\mathcal{S}} L, \phi \in \mathcal{C}_{\mathcal{S}} M$,

$$\mathcal{C}_{\mathcal{S}} h(\theta) \subseteq \phi \iff \theta \subseteq (h \times h)^{-1}(\phi).$$

Theorem 4.2 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *The map h is closed iff h has a right adjoint, r, and for all $x \in L, m \in M$,*

$$r(h(x) \vee m) = x \vee r(m).$$

(b) *The map h is open iff h has a left adjoint, l, and for all $x \in L, m \in M$,*

$$l(h(x) \wedge m) = x \wedge l(m).$$

Theorem 4.3 *Let L be an $\mathcal{S}$-frame and θ an $\mathcal{S}$-congruence on L.*
(a) The quotient map $q : L \to L/\theta$ is closed if and only if θ is a closed $\mathcal{S}$-congruence; i.e. $\theta = \nabla_a$ for some $a \in L$.
(b) The quotient map $q : L \to L/\theta$ is open if and only if θ is an open $\mathcal{S}$-congruence; that is, $\theta = \Delta_a$ for some $a \in L$.

Definition 4.4 *Let $h : L \to M$ be an $\mathcal{S}$-frame map. We say that h is* dense *(resp.,* codense*) if for all $a \in L, h(a) = 0$ (resp., $h(a) = 1$) implies that $a = 0$ (resp., $a = 1$).*

Lemma 4.5 *Let $h : L \to M$ be an $\mathcal{S}$-frame map.*

(a) *If h is dense and closed, then h is one-one. If h is dense, closed and onto, then h is an isomorphism.*

(b) *If h is codense and open, then h is one-one. If h is codense, open and onto, then h is an isomorphism.*

Lemma 4.6 *Suppose that $f : L \to M$ and $g : M \to N$ are $\mathcal{S}$-frame maps.*

(a) *(i) If f and g are both closed, then $g \circ f$ is closed.*
     (ii) If $g \circ f$ is closed and g is one-one, then f is closed.
     (iii) If $g \circ f$ is closed and f is onto, then g is closed.

(b) *As above but replace "closed" by "open".*

5   Boolean properties for partial frames

The material in this section comes from [16].

We begin by recalling how matters stand in the case of full frames. A Boolean frame is simply a frame that is also a Boolean algebra, that is, every element has a complement. However, Booleanness can also be characterized in a different way. For any frame M, let $M_{**} = \{x^{**} : x \in M\}$ where $x^* = \bigvee \{z \in M : z \wedge x = 0\}$, the pseudocomplement of x. The frame map $p : M \to M_{**}$ given by $p(x) = x^{**}$ is called the Booleanization of M. It is the least dense quotient of M, but is also the unique dense Boolean quotient of M. A frame is then Boolean if and only if it is isomorphic to its Booleanization.

Following Madden's lead in [19], in [11] we constructed a least dense quotient for partial frames. The codomain need not be Boolean, however, as Madden already noted in the case of κ-frames. We use his terminology, "d-reduced", to refer to those partial frames isomorphic to their least dense quotients. We refer the reader to Definition 2.3(l) for our notation and terminology in this regard.

The next result characterizes those $\mathcal{S}$-frames, L, that are Boolean algebras, in several ways. These involve the free frame over L, the congruence frame of L and the the relationship between these two entities.

Proposition 5.1 *Let L be an $\mathcal{S}$-frame. The following are equivalent:*

(i) *L is Boolean; that is, every element of L is complemented.*

(ii) *All principal $\mathcal{S}$-ideals in $\mathcal{H}_{\mathcal{S}}L$ are complemented.*

(iii) *The embedding $e : \mathcal{H}_{\mathcal{S}}L \to \mathcal{C}_{\mathcal{S}}L$ is an isomorphism.*

(iv) *Every $\mathcal{S}$-congruence θ of L is an arbitrary join of $\mathcal{S}$-congruences of the form ∇_a, for some $a \in L$.*

In our experience with partial frames, it has often proved useful to compare properties for a partial frame with the analogous properties for the corresponding free frame. We do this now for Booleanness.

We recall that, if M is a frame and $x \in M$, we call x a *dense* element of M if $x^* = 0$.

Proposition 5.2 *Let L be an $\mathcal{S}$-frame. The following are equivalent:*

(i) *The frame $\mathcal{H}_{\mathcal{S}}L$ is Boolean.*

(ii) *$\downarrow 1$ is the only dense element of $\mathcal{H}_{\mathcal{S}}L$.*

(iii) *The $\mathcal{S}$-frame embedding $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ is an isomorphism.*

(iv) *Every $\theta \in \mathcal{C}_{\mathcal{S}}L$ has the form $\theta = \nabla_a$, for some $a \in L$.*

We now provide four provably distinct conditions akin to Booleanness for partial frames. In the setting of (full) frames they all amount to every element being complemented.

Theorem 5.3 *Let L be an $\mathcal{S}$-frame. In the following list of conditions, each one implies the succeeding one, but not conversely.*

(i) *$\mathcal{H}_{\mathcal{S}}L$ is a Boolean frame.*

(ii) *L is a Boolean frame.*

(iii) *L is a Boolean $\mathcal{S}$-frame.*

(iv) *L is a d-reduced $\mathcal{S}$-frame.*

Proof. (a)$\Rightarrow$(b): ($\not\Leftarrow$) See Example 5.5.
(b)$\Rightarrow$(c): ($\not\Leftarrow$): See Example 5.6.
(c)$\Rightarrow$(d): ($\not\Leftarrow$): See Example 5.4.

Example 5.4 Let $\mathcal{S}$ designate countable subsets, and consider the σ-frame $L = \mathcal{P}_C(\mathbb{R})$, which consists of all countable subsets of $\mathbb{R}$ together with $\mathbb{R}$ as the top element. Countable join is union, binary meet is intersection.
Here $(X, Y) \in \chi_0$ if and only if, for any countable subset U, $U \cap X = \emptyset \iff U \cap Y = \emptyset$, which makes $X = Y$. So $\chi_0 = \Delta$, which makes $\mathcal{P}_C(\mathbb{R})$ d-reduced. However, $\mathcal{P}_C(\mathbb{R})$ is clearly not Boolean.

Example 5.5 Let $\mathcal{S}$ designate countable subsets, and let $\mathcal{L}$ consist of all subsets of $\mathbb{R}$. Clearly $\mathcal{L}$ is a Boolean frame. We show that $\mathcal{H}_{\mathcal{S}}\mathcal{L}$ is not Boolean, using condition (d) of Proposition 5.2.
Let $\mathcal{I} = \{X \subseteq \mathbb{R} : X \cap (\mathbb{R} \backslash \mathbb{Q})$ is countable$\}$. Then $\mathcal{I}$ is a σ-ideal of $\mathcal{L}$; that is, a downset closed under countable unions. By Definition 2.3(g)(iv), $\bigvee_{X \in \mathcal{I}} \nabla_X = \bigcup_{X \in \mathcal{I}} \nabla_X$ and this cannot have the form ∇_Z for any $Z \in \mathcal{L}$, since that would require $Z \supseteq X$ for all $X \in \mathcal{I}$, and hence $Z = \mathbb{R}$; a contradiction.

Example 5.6 Let L consist of all countable and co-countable subsets of the real line, and let $\mathcal{S}$ designate countable subsets. Clearly L is a Boolean σ-frame, but not a complete lattice, so not a frame.
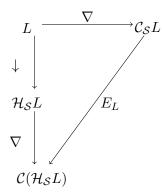
6   Comparing congruences on a partial frame and its free frame

The material in this section comes from [16].
In this section, for a partial frame L, we compare $\mathcal{C}_{\mathcal{S}}L$, the frame of $\mathcal{S}$-congruences of L, with $\mathcal{C}(\mathcal{H}_{\mathcal{S}}L)$, the frame of (frame) congruences on $\mathcal{H}_{\mathcal{S}}L$, the free frame over L. The universal property of the embedding $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ provides a frame map $E_L : \mathcal{C}_{\mathcal{S}}L \to \mathcal{C}(\mathcal{H}_{\mathcal{S}}L)$. We give an explicit description of this map, and show that it provides a natural transformation.
We then turn our attention to its right adjoint $D_L : \mathcal{C}(\mathcal{H}_{\mathcal{S}}L) \to \mathcal{C}_{\mathcal{S}}L$. Again, we provide an explicit description of this function, including an interesting and useful action on closed congruences (Lemma 6.8).

Definition 6.1 Let L be an $\mathcal{S}$-frame. Consider this diagram

By the universal property of $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ there exists a unique frame map $E_L : \mathcal{C}_{\mathcal{S}}L \to \mathcal{C}(\mathcal{H}_{\mathcal{S}}L)$ such that $E_L \circ \nabla = \nabla \circ \downarrow$; that is, for all $a \in L$
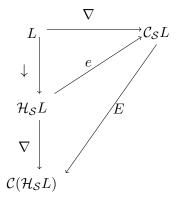
$$E_L(\nabla_a) = \nabla_{\downarrow a}.$$

Lemma 6.2 *Let L be an $\mathcal{S}$-frame.*

(i) *For $\theta \in \mathcal{C}_{\mathcal{S}}L$, $E(\theta)$ is the frame congruence on $\mathcal{H}_{\mathcal{S}}L$ generated by $\{(\downarrow x, \downarrow y) : (x,y) \in \theta\}$; this is denoted by $E(\theta) = \langle (\downarrow x, \downarrow y) : (x,y) \in \theta \rangle$.*

(ii) *The frame map $E : \mathcal{C}_{\mathcal{S}}L \to \mathcal{C}(\mathcal{H}_{\mathcal{S}}L)$ is dense.*

Corollary 6.3 *Let L be an $\mathcal{S}$-frame and $E : \mathcal{C}_{\mathcal{S}}L \to \mathcal{C}(\mathcal{H}_{\mathcal{S}}L)$ given as in Definition 6.1. For all $a \in L$:*

(i) $E(\nabla_a) = \nabla_{\downarrow a}$

(ii) $E(\Delta_a) = \Delta_{\downarrow a}$

Remark 6.4 Let L be an $\mathcal{S}$-frame. the embedding $e : \mathcal{H}_{\mathcal{S}}L \to \mathcal{C}_{\mathcal{S}}L$ of Definition 2.4 can be incorporated into the diagram of Definition 6.1 as follows:

Note that

- the upper triangle commutes, since $e \circ \downarrow = \nabla$.
- the lower triangle commutes, since, for $I \in \mathcal{H}_{\mathcal{S}}L$,
  $E \circ e(I) = E(\bigvee_{i \in I} \nabla_i) = \bigvee_{i \in I} E(\nabla_i) = \bigvee_{i \in I} \nabla_{\downarrow i} = \nabla_I.$
  Alternatively, this can be seen because the outer diagram commutes and every $\mathcal{S}$-ideal is generated by principal $\mathcal{S}$-ideals.

Proposition 6.5 *The function E_L provides a natural transformation from the functor $\mathcal{C}_{\mathcal{S}}$ to the functor $\mathcal{C}\mathcal{H}_{\mathcal{S}}$.*

We now define, for any $\mathcal{S}$-frame L, the function D_L. In a subsequent lemma, D_L is seen to be the right

adjoint of the frame map E_L.

Definition 6.6 Let L be an $\mathcal{S}$-frame, and Φ a frame congruence on the frame $\mathcal{H}_\mathcal{S}L$. Define

$$D_L(\Phi) = \{(x, y) \in L \times L : (\downarrow x, \downarrow y) \in \Phi\}.$$

Lemma 6.7 *Let L be an $\mathcal{S}$-frame.*

 (i) *For any frame congruence Φ on $\mathcal{H}_\mathcal{S}L$, $D_L(\Phi)$ is an $\mathcal{S}$-congruence on L.*

 (ii) *The function $D_L : \mathcal{C}(\mathcal{H}_\mathcal{S}L) \to \mathcal{C}_\mathcal{S}L$ is the right adjoint of the frame map $E_L : \mathcal{C}_\mathcal{S}L \to \mathcal{C}(\mathcal{H}_\mathcal{S}L)$ of Definition 6.1.*

(iii) *The function D_L preserves bottom, top and arbitrary meets.*

We now provide further properties of D, including its action on certain special congruences. We note that the proof of Lemma 6.8(a) uses the fact that, for I an $\mathcal{S}$-ideal of an $\mathcal{S}$-frame L, $\bigvee_{i \in I} \nabla_i = \bigcup_{i \in I} \nabla_i$. This is not immediately obvious, but was proved in [15] Lemma 3.1.

Lemma 6.8 *Let L be an $\mathcal{S}$-frame, and D as in Definition 6.6.*

 (i) *For all $I \in \mathcal{H}_\mathcal{S}L$, $D(\nabla_I) = \bigcup_{i \in I} \nabla_i$.*

 (ii) *For all $a \in L$,*
    (a) $D(\nabla_{\downarrow a}) = \nabla_a$
    (b) $D(\Delta_{\downarrow a}) = \Delta_a$

(iii) *For $I \in \mathcal{H}_\mathcal{S}L$, I is principal $\iff D(\nabla_I) \vee D(\Delta_I) = \nabla$.*

Definition 6.9 Let M be a full frame. For any $a \in M$ we say a is an $\mathcal{S}$-*Lindelöf element* of M if the following condition holds:
    If $a = \bigvee B$ for some $B \subseteq M$, then $a = \bigvee D$ for some designated subset D of M such that $D \subseteq B$.

See [7] for details about this notion. In particular, Lemma 4.3 of that paper characterizes the $\mathcal{S}$-Lindelöf elements of $\mathcal{H}_\mathcal{S}L$ as being the principal $\mathcal{S}$-ideals.

The next result characterizes those rather special $\mathcal{S}$-frames L for which E_L is an isomorphism.

Theorem 6.10 *Let L be an $\mathcal{S}$-frame. The following are equivalent:*

 (i) *The embedding $\downarrow : L \to \mathcal{H}_\mathcal{S}L$ is an isomorphism.*

 (ii) *Every $\mathcal{S}$-ideal of L is principal.*

(iii) *L is a frame and every element of L is $\mathcal{S}$-Lindelöf.*

(iv) *The frame map $E : \mathcal{C}_\mathcal{S}L \to \mathcal{C}(\mathcal{H}_\mathcal{S}L)$ is an isomorphism.*

The equivalent conditions of Theorem 6.10 might seem rather strong. Here are some examples which show that these can obtain.

Example 6.11 The conditions of Theorem 6.10 hold in the following examples:

• $\mathcal{S}$ selects finite subsets and L is a finite frame.

• $\mathcal{S}$ selects countable subsets, and L consists of the open subsets of the real line.

• $\mathcal{S}$ selects finite subsets, or $\mathcal{S}$ selects countable subsets, and L consists of the cofinite subsets of the real line, together with the empty set.

7   Closed and open embeddings into the free frame and the congruence frame

Theorem 7.1 *Let L be an $\mathcal{S}$-frame and $\downarrow : L \to \mathcal{H}_\mathcal{S}L$ the embedding into its free frame.*

 (a) *The map $\downarrow$ has a right adjoint iff $\downarrow$ is an isomorphism.*

(b) *The map ↓ is closed iff ↓ is an isomorphism.*

(c) *The map ↓ has a left adjoint iff L is a complete lattice.*

(d) *The map ↓ is open iff L is a frame.*

Theorem 7.2 *Let L be an $\mathcal{S}$-frame and $\nabla : L \to \mathcal{C}_{\mathcal{S}}L$ the embedding into its congruence frame.*

(a) *The map ∇ is closed iff ∇ is an isomorphism.*

(b) *The map ∇ is open iff L is a Boolean frame.*

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