

Representations of Domains via CF-approximation Spaces

Guojun Wu^{1,2} Luoshan Xu^{1,3}

*College of Mathematical Science
Yangzhou University
Yangzhou 225002, P. R. China*

Abstract

Representations of domains mean in a general way representing a domain as a suitable family endowed with set-inclusion order of some mathematical structures. In this paper, representations of domains via CF-approximation spaces are considered. Concepts of CF-approximation spaces and CF-closed sets are introduced. It is proved that the family of CF-closed sets in a CF-approximation space endowed with set-inclusion order is a continuous domain and that every continuous domain is isomorphic to the family of CF-closed sets of some CF-approximation space endowed with set-inclusion order. The concept of CF-approximable relations is introduced using a categorical approach, which later facilitates the proof that the category of CF-approximation spaces and CF-approximable relations is equivalent to that of continuous domains and Scott continuous maps.

Keywords: CF-approximation space; CF-closed set; CF-approximable relation; continuous domain; abstract base

1 Introduction

Domain theory is one of the important research fields of theoretical computer science [2]. In recent years of research in domain theory, there is a growing body of scholarly work towards synthesizing various mathematical fields such as ordered structures, topological spaces, formal contexts, rough sets, and various kinds of logic. One of such syntheses is to create representation for various kinds of domains using abstract bases [8,12], formal topologies [13], information systems [10,12], formal contexts [4]-[11], and so on. Amongst these, representation via abstract bases appears to be most natural due to its simplicity.

By representation of domains, we mean any general way by which one can characterize a domain using a suitable family of some mathematical structures ordered by the set-theoretic inclusion. With this understanding, clearly, every continuous domain can be represented by c-infs [10], abstract bases, formal topologies [13], etc. Recently, Qingguo Li, et. al. in [11] introduced attribute continuous formal contexts which are quadruples, and showed that every continuous domain can be represented by attribute continuous formal contexts.

¹ Supported by National Natural Science Foundation of China (11671008).

² Email: 2386858989@qq.com

³ Email: luoshanxu@hotmail.com

While representation using abstract bases appears to be the most natural and simple, its scope of study is unfortunately too narrow in that it is easy to miss out on something deeper. Noticing from rough set theory [7] that abstract bases are all special generalized approximation spaces (GA-spaces, for short) [14], we consider generalize an abstract base to a CF-approximation space which is a GA-space with some coordinating family of finite sets. Since the lower approximation operator \underline{R} and the upper approximation operator \overline{R} are mutually dual in a CF-approximation space, we mainly use the upper approximation operator \overline{R} and introduce CF-closed sets which are generalizations of round ideals in abstract bases. With these concepts, representations of domains via CF-approximation spaces are obtained. We will see that this approach of representing domains is more general than the approach of representing domains by abstract bases. We also introduce the concept of CF-approximable relations using a categorical approach and prove that the category of CF-approximation spaces and CF-approximable relations is equivalent to that of continuous domains and Scott continuous maps. This work makes links naturally between domains and rough sets.

2 Preliminaries

We quickly recall some basic notions and results of domain theory. For a set U and $X \subseteq U$, we use $\mathcal{P}(U)$ to denote the power set of U , $\mathcal{P}_{fin}(U)$ to denote the family of all nonempty finite subsets of U and X^c to denote the complement of X in U . The symbol $F \subseteq_{fin} X$ means F is a finite subset of X . For notions which we do not explicitly define herein, the reader may refer to [2,3].

Let (L, \leq) be a poset. A *principal ideal* (resp., *principal filter*) of L is a set of the form $\downarrow x = \{y \in L \mid y \leq x\}$ (resp., $\uparrow x = \{y \in L \mid x \leq y\}$). For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists x \in A, y \leq x\}$ and $\uparrow A = \{y \in L \mid \exists x \in A, x \leq y\}$. A subset A is a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a *lower bound* (resp., an *upper bound*) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). The supremum of A is the least upper bound of A , denoted by $\bigvee A$ or $\sup A$. The infimum of A is the greatest lower bound of A , denoted by $\bigwedge A$ or $\inf A$. A nonempty subset D of L is *directed* if every finite subset of D has an upper bound in D . A subset C of L is *consistent* if C has an upper bound in L . A poset L is a *directed complete partially ordered set* (*dcpo*, for short) if every directed subset of L has a supremum. A *semilattice* (resp., *sup-semilattice*) is a poset in which every pair of elements has an infimum (resp., a supremum). A *complete lattice* is a poset in which every subset has a supremum (equivlently, has an infimum). If any finite consistent subset A of L has a supremum, then L is called a *cusl*. If any consistent subset B of L has a supremum, then L is called a *bc-poset*.

Let L be a semilattice and $K \subseteq L$. If for all $x, y \in K$ it holds that $x \wedge y \in K$, then K is called a *subsemilattice* of L .

Recall that in a poset P , we say that x *way-below* y , written $x \ll y$ if whenever D is a directed set that has a supremum with $\sup D \geq y$, then $x \leq d$ for some $d \in D$. If $x \ll x$, then x is called a *compact element* of P , the set $\{x \in P \mid x \ll x\}$ is denoted by $K(P)$. The set $\{y \in P \mid x \ll y\}$ will be denoted by $\uparrow x$ and $\{y \in P \mid y \ll x\}$ denoted by $\downarrow x$. A poset P is said to be *continuous* (resp., *algebraic*) if for all $x \in P$, $\downarrow x$ is directed (resp., $\downarrow x \cap K(P)$ is directed) and $x = \bigvee \downarrow x$ (resp., $x = \bigvee (\downarrow x \cap K(P))$). If a dcpo P is continuous (resp., algebraic), then P is called a *continuous domain* (resp., *an algebraic domain*). If a continuous domain P is a semilattice (resp., sup-semilattice, complete lattice), then P is called a *continuous semilattice* (resp., *continuous sup-semilattice*, *continuous lattice*). If a bc-poset P is also a continuous domain, then P is called a *bc-domain*. If an algebraic domain L is a semilattice and $K(L)$ is a subsemilattice of L , then L is called *arithmetic semmilattice*.

Let L and P be dcpos, and $f : L \rightarrow P$ a map. If for any directed subset $D \subseteq L$, $f(\bigvee D) = \bigvee f(D)$, then f is called a *Scott continuous map*.

Lemma 2.1 ([2]) *Let P be a poset. Then for all $x, y, u, z \in P$,*

- (1) $x \ll y \Rightarrow x \leq y$;
- (2) $u \leq x \ll y \leq z \Rightarrow u \ll z$.

Lemma 2.2 *If P is a continuous poset, then the way-below relation \ll has the interpolation property:*

$$x \ll z \Rightarrow \exists y \in P \text{ such that } x \ll y \ll z.$$

Definition 2.3 *Let P be a poset, $B \subseteq P$. The set B is called a basis for P if for all $a \in P$, there is a directed set $D_a \subseteq B \cap \downarrow a$ such that $\sup_P D_a = a$, where the subscripted P indicates that the operation (in this case, the supremum) is taken in poset P .*

It is well known that a poset P is continuous iff it has a basis and that P is algebraic iff $K(P)$ is a basis.

A binary relation $R \subseteq U \times U$ on a set U is called *transitive* if xRy and yRz implies xRz for all $x, y, z \in U$. A binary relation R is said a *preorder* if it is reflexive and transitive.

Definition 2.4 (see [2,3]) *Let (U, \prec) be a set equipped with a binary relation. The binary relation \prec is called fully transitive if it is transitive and satisfies the strong interpolation property:*

$$\forall |F| < \infty, F \prec z \Rightarrow \exists y \prec z \text{ such that } F \prec y,$$

where $F \prec y$ means for all $t \in F$, $t \prec y$. If (B, \prec) is a set equipped with a binary relation which is fully transitive, then (B, \prec) is called an *abstract basis*.

Definition 2.5 ([2,3]) *Let (B, \prec) be an abstract basis. A non-empty subset I of B is a round ideal if*

- (1) $\forall y \in I, x \prec y \Rightarrow x \in I$;
- (2) $\forall x, y \in I, \exists z \in I$ such that $x \prec z$ and $y \prec z$.

All the round ideals of B in set-inclusion order is called the *round ideal completion* of B , denoted by $RI(B)$.

Observe that if B is a basis for a continuous domain P , then (B, \ll) , the restriction of the way-below relation to B , is an abstract basis. And it is known (see in [10]) that P in this case is isomorphic to $RI(B)$.

Proposition 2.6 *If P is a continuous domain, then (P, \ll) is an abstract basis and $RI(P, \ll) \cong (P, \leq)$.*

Next, we introduce some terminologies imported from rough set theory. A set U with a binary relation R is called a *generalized approximation space* (*GA-space*, for short). Let (U, R) be a GA-space. Define $R_s, R_p : U \rightarrow \mathcal{P}(U)$ such that for all $x \in U$, $R_s(x) = \{y \in U \mid xRy\}$, $R_p(x) = \{y \in U \mid yRx\}$.

Lower and upper approximation operators are key notions in GA-spaces.

Definition 2.7 (cf. [17]) *Let (U, R) be a GA-space. For $A \subseteq U$, define*

$$\underline{R}(A) = \{x \in U \mid R_s(x) \subseteq A\}, \quad \overline{R}(A) = \{x \in U \mid R_s(x) \cap A \neq \emptyset\}.$$

The operators $\underline{R}, \overline{R} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ are respectively called the *lower and upper approximation operators* in (U, R) , where $\mathcal{P}(U)$ is the power set of U .

Lemma 2.8 (cf. [9]) *Let (U, R) be a GA-space. Then the lower and upper approximation operators \underline{R} and \overline{R} have the following properties.*

- (1) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$, where A^c is the complement of $A \subseteq U$.
- (2) $\underline{R}(U) = U$, $\overline{R}(\emptyset) = \emptyset$.
- (3) Let $\{A_i \mid i \in I\} \subseteq \mathcal{P}(U)$. Then $\underline{R}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \underline{R}(A_i)$, $\overline{R}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} \overline{R}(A_i)$.
- (4) If $A \subseteq B \subseteq U$, then $\underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.
- (5) For all $x \in U$, $\overline{R}(\{x\}) = R_p(x)$.

Lemma 2.9 [16] *Let (U, R) be a GA-space. Then R is reflexive iff for all $X \subseteq U$, $X \subseteq \overline{R}(X)$.*

Lemma 2.10 [16] *Let (U, R) be a GA-space. Then R is transitive iff for all $X \subseteq U$, $\overline{R}(\overline{R}(X)) \subseteq \overline{R}(X)$.*

Definition 2.11 ([15]) *Let (U, R) be a GA-space and $A \subseteq U$. The set A is called R -open if $A \subseteq \underline{R}(A)$ and R -closed if $\overline{R}(B) \subseteq B$.*

For a preorder R , the operator \underline{R} is an interior operator of a topology, so we have

Definition 2.12 ([15]) *If R is a preorder, then GA-space (U, R) is called a topological GA-space.*

The next proposition shows that all R -open sets of (U, R) form a topology on U .

Proposition 2.13 *Let (U, R) be a GA-space. Then $\tau_R = \{A \subseteq U \mid A \subseteq \underline{R}(A)\}$ is an Alexandrov topology on U .*

Proof. Obviously, $\emptyset, U \in \tau_R$. By Lemma 2.8(3), we have τ_R is closed under arbitrary intersections. Let $A_i \in \tau_R$, namely, $A_i \subseteq \underline{R}(A_i)$ ($i \in I$). Then $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} (\underline{R}(A_i))$. It follows from $\underline{R}(A_i) \subseteq \underline{R}(\bigcup_{i \in I} A_i)$ that $\bigcup_{i \in I} (\underline{R}(A_i)) \subseteq \underline{R}(\bigcup_{i \in I} A_i)$. Then $\bigcup_{i \in I} A_i \subseteq \underline{R}(\bigcup_{i \in I} A_i)$. So $\bigcup_{i \in I} A_i \in \tau_R$, namely, τ_R is closed under arbitrary unions. This shows that τ_R is an Alexandrov topology. \square

The above topology τ_R is called a *topology induced by relation R* . Obviously, all the R -closed sets of (U, R) are precisely all the closed sets of τ_R .

3 CF-approximation Spaces and CF-closed Sets

For an abstract base (B, \prec) , we naturally have the triple $(B, \prec, \{\{b\} \mid b \in B\})$ and that the family $\{\downarrow^\prec b \mid b \in B\}$ is a base of the continuous domain $RI(B)$, where $\downarrow^\prec b = \{c \in B \mid c \prec b\}$. We generalize an abstract base to a GA-space with consistent family of finite subsets (CF-approximation space, for short) by changing (B, \prec) to a GA-space (U, R) with R being transitive and changing the family $\{\{b\} \mid b \in B\}$ to a suitable family \mathcal{F} of some finite subsets of U . We hope that the family \mathcal{F} can also induce a base of a continuous domain.

Definition 3.1 *Let (U, R) be a GA-space, R a transitive relation and $\mathcal{F} \subseteq \mathcal{P}_{fin}(U) \cup \{\emptyset\}$. If for all $F \in \mathcal{F}$, whenever $K \subseteq_{fin} \overline{R}(F)$, there always exists $G \in \mathcal{F}$ such that $K \subseteq \overline{R}(G)$ and $G \subseteq \overline{R}(F)$, then (U, R, \mathcal{F}) is called a generalized approximation space with consistent family of finite subsets, or a CF-approximation space, for short.*

Lemma 3.2 *Let (U, R) be a GA-space, $A, B \subseteq U$. If R is a transitive relation, then $\overline{R}(B) \subseteq \overline{R}(A)$ when $B \subseteq \overline{R}(A)$.*

Proof. It follows from Lemma 2.8(4) and 2.10. \square

Definition 3.3 *Let (U, R, \mathcal{F}) be a CF-approximation space, $E \subseteq U$. If for all $K \subseteq_{fin} E$, there always exists $F \in \mathcal{F}$ such that $K \subseteq \overline{R}(F) \subseteq E$ and $F \subseteq E$, then E is called a CF-closed set of (U, R, \mathcal{F}) . The collection of all CF-closed sets of (U, R, \mathcal{F}) is denoted by $\mathfrak{C}(U, R, \mathcal{F})$.*

Remark 3.4 (1) *If $\emptyset \in \mathfrak{C}(U, R, \mathcal{F})$, then $\emptyset \in \mathcal{F}$ by $\overline{R}(\emptyset) = \emptyset$.*

(2) *For CF-approximation space (U, R, \mathcal{F}) , if $\mathcal{F} = \{\{x\} \mid x \in U\}$, then (U, R) is an abstract base by Lemma 2.8(5), and all the CF-closed sets of (U, R, \mathcal{F}) are precisely all the round ideals of (U, R) .*

The following example shows that (U, R) is not necessarily an abstract base when (U, R, \mathcal{F}) is a CF-approximation space, showing that CF-approximation spaces is a generalization of abstract bases.

Example 3.5 *Let $U = \mathbb{N}$, $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (4, 3)\} \cup \{(i, i) \mid i \geq 5\}$, $\mathcal{F} = \{\{1\}, \{1, 2\}, \emptyset\} \cup \{\{i\} \mid i \geq 5\}$. It is easy to check that (U, R, \mathcal{F}) is a CF-approximation space and $\mathfrak{C}(U, R, \mathcal{F}) = \{\emptyset, \{1\}\} \cup \{\{i\} \mid i \geq 5\}$ is a continuous domain. Notice that $(1, 4), (2, 4) \in R$, but there is no $t \in U$ such that $(1, t), (2, t), (t, 4) \in R$. So (U, R) is not an abstract base.*

Proposition 3.6 *Let (U, R, \mathcal{F}) be a CF-approximation space. If $E \in \mathfrak{C}(U, R, \mathcal{F})$, then E is an R -closed set.*

Proof. If $x \in \overline{R}(E)$, then $R_s(x) \cap E \neq \emptyset$. So there is $y \in U$ such that xRy and $y \in E$. By Definition 3.3, there exists $F \in \mathcal{F}$, such that $y \in \overline{R}(F) \subseteq E$ and $F \subseteq E$. By the transitivity of R , we know that $\overline{R}(\{y\}) \subseteq \overline{R}(\overline{R}(F)) \subseteq \overline{R}(F) \subseteq E$. Thus $\overline{R}(\{y\}) \subseteq E$. It is clear that $x \in \overline{R}(\{y\}) \subseteq E$ because of xRy . By the arbitrariness of $x \in \overline{R}(E)$, we know that $\overline{R}(E) \subseteq E$. This shows that E is an R -closed set. \square

Proposition 3.7 *Let (U, R, \mathcal{F}) be a CF-approximation space, then the following statements hold:*

- (1) *For any $F \in \mathcal{F}$, $\overline{R}(F) \in \mathfrak{C}(U, R, \mathcal{F})$;*
- (2) *If $E \in \mathfrak{C}(U, R, \mathcal{F})$, $A \subseteq E$, then $\overline{R}(A) \subseteq E$;*
- (3) *If $\{E_i\}_{i \in I} \subseteq \mathfrak{C}(U, R, \mathcal{F})$ is a directed family, then $\bigcup_{i \in I} E_i \in \mathfrak{C}(U, R, \mathcal{F})$.*

Proof. (1) Follows directly by Definition 3.1 and the transitivity of R .

(2) Follows from $A \subseteq E$ and Lemma 2.8(4) that $\overline{R}(A) \subseteq \overline{R}(E)$. By Proposition 3.6, we know that $\overline{R}(E) \subseteq E$. Thus $\overline{R}(A) \subseteq E$.

(3) Follows directly from Definition 3.3. \square

The proposition above shows that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a dcpo. The following proposition gives equivalent characterizations of CF-closed sets.

Proposition 3.8 *The following statements are equivalent for a CF-approximation space (U, R, \mathcal{F}) :*

- (1) $E \in \mathfrak{C}(U, R, \mathcal{F})$;
- (2) *The family $\mathcal{A} = \{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E\}$ is directed and $E = \bigcup \mathcal{A}$;*
- (3) *There exists a family $\{F_i\}_{i \in I} \subseteq \mathcal{F}$ such that $\{\overline{R}(F_i)\}_{i \in I}$ is directed, and $E = \bigcup_{i \in I} \overline{R}(F_i)$;*
- (4) *There always exists $F \in \mathcal{F}$ such that $K \subseteq \overline{R}(F) \subseteq E$ whenever $K \subseteq_{fin} E$.*

Proof. If $E = \emptyset \in \mathfrak{C}(U, R, \mathcal{F})$, the proposition holds obviously. Let $E \neq \emptyset$.

(1) \Rightarrow (2) By Definition 3.3, we know that \mathcal{A} is not empty. Let $X_1, X_2 \in \mathcal{A}$, then there exist $F_1, F_2 \in \mathcal{F}$ and $F_1, F_2 \subseteq E$, such that $X_1 = \overline{R}(F_1)$, $X_2 = \overline{R}(F_2)$. By $F_1 \cup F_2 \subseteq_{fin} E$ and Definition 3.3 we know that there exists $F_3 \in \mathcal{F}$, such that $F_1 \cup F_2 \subseteq \overline{R}(F_3)$ and $F_3 \subseteq E$. By the transitivity of R and Lemma 3.2 we know that $\overline{R}(F_1) \subseteq \overline{R}(F_3)$, $\overline{R}(F_2) \subseteq \overline{R}(F_3)$. This shows that \mathcal{A} is directed. Next we prove $E = \bigcup \mathcal{A}$. By Proposition 3.7(2) we know that $\bigcup \mathcal{A} \subseteq E$ holds. Conversely, if $x \in E$, then by Definition 3.3, there is $F \in \mathcal{F}$ such that $x \in \overline{R}(F) \subseteq E$ and $F \subseteq E$. So $x \in \bigcup \mathcal{A}$. By the arbitrariness of $x \in E$ we know that $E \subseteq \bigcup \mathcal{A}$. Thus $E = \bigcup \mathcal{A}$.

(2) \Rightarrow (3) Trivial.

(3) \Rightarrow (4) Follows directly from the finiteness of K and the directedness of $\{\overline{R}(F_i)\}_{i \in I}$.

(4) \Rightarrow (1) If $K \subseteq_{fin} E$, then there exists $F \in \mathcal{F}$ such that $K \subseteq \overline{R}(F) \subseteq E$. By Definition 3.1, there exists $G \in \mathcal{F}$ such that $K \subseteq \overline{R}(G)$ and $G \subseteq \overline{R}(F)$. By Lemma 3.2 we know that $\overline{R}(G) \subseteq \overline{R}(F) \subseteq E$. Thus $K \subseteq \overline{R}(G) \subseteq E$. Noticing that $G \subseteq \overline{R}(F) \subseteq E$, by Definition 3.3 we obtain that $E \in \mathfrak{C}(U, R, \mathcal{F})$. \square

The following theorem characterizes the way-below relation \ll in dcpo $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.

Theorem 3.9 *Let (U, R, \mathcal{F}) be a CF-approximation space, $E_1, E_2 \in \mathfrak{C}(U, R, \mathcal{F})$. Then $E_1 \ll E_2$ if and only if there exists $F \in \mathcal{F}$ such that $E_1 \subseteq \overline{R}(F)$ and $F \subseteq E_2$.*

Proof. \Rightarrow : It follows from $E_2 \in \mathfrak{C}(U, R, \mathcal{F})$ and Proposition 3.8(2) that $E_2 = \bigcup \{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E_2\}$ and that $\{\overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E_2\}$ is directed. If $E_1 \ll E_2$, then there exists $F \in \mathcal{F}$ such that $F \subseteq E_2$, $E_1 \subseteq \overline{R}(F)$.

\Leftarrow : For any directed family $\{C_i\}_{i \in I} \subseteq \mathfrak{C}(U, R, \mathcal{F})$, if $E_2 \subseteq \bigvee_{i \in I} C_i = \bigcup_{i \in I} C_i$, then by $F \subseteq E_2$ and the finiteness of F we know that there exists $i_0 \in I$ such that $F \subseteq C_{i_0}$. By Proposition 3.7(2) and $E_1 \subseteq \overline{R}(F)$, we know that $E_1 \subseteq \overline{R}(F) \subseteq C_{i_0}$, showing that $E_1 \ll E_2$. \square

Corollary 3.10 *Let (U, R, \mathcal{F}) be a CF-approximation space, $E \in \mathfrak{C}(U, R, \mathcal{F})$, $F \in \mathcal{F}$. The following statements hold:*

- (1) *If $F \subseteq E$, then $\overline{R}(F) \ll E$;*
- (2) *$\overline{R}(F) \ll \overline{R}(F)$ if and only if there exists $G \in \mathcal{F}$, such that $G \subseteq \overline{R}(G) = \overline{R}(F)$.*

Proof. Follows directly from Theorem 3.9. \square

Theorem 3.11 *Let (U, R, \mathcal{F}) be a CF-approximation space. Then $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a continuous domain.*

Proof. By Proposition 3.7 we see that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a dcpo. Set $\mathcal{B} = \{\overline{R}(F) \mid F \in \mathcal{F}\}$. Then \mathcal{B} is a base of $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ by Proposition 3.8(2) and Corollary 3.10(1). Thus $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is a continuous domain. \square

The following theorem shows that any continuous domain (L, \leq) can induce a CF-approximation space.

Theorem 3.12 *Let (L, \leq) be a continuous domain, R_L the way-below relation “ \ll ” of (L, \leq) ; $\mathcal{F}_L = \{F \subseteq_{fin} L \mid F \text{ has a top element}\}$. For any $F \in \mathcal{F}_L$, let c_F be the top element of F . Then (L, R_L, \mathcal{F}_L) is a CF-approximation space.*

Proof. By Lemma 2.1, we know that $R_L = \ll$ is transitive. For any $F \in \mathcal{F}_L$, by Lemma 2.8(5), we have that $\overline{R}_L(F) = \downarrow c_F$. For $K \subseteq_{fin} \overline{R}_L(F) = \downarrow c_F$, by that L is a continuous domain, we know that $\downarrow c_F$ is directed. Then there exists $x \in \downarrow c_F$ such that $K \subseteq \downarrow x$. It follows from $x \ll c_F$ and Lemma 2.2 that there is $y \in L$ such that $x \ll y \ll c_F$. Thus $K \subseteq \downarrow y$. Set $G = \{y\} \in \mathcal{F}_L$. By that $K \subseteq \overline{R}_L(G) = \downarrow y$ and $G \subseteq \overline{R}_L(F) = \downarrow c_F$, we have that (L, R_L, \mathcal{F}_L) is a CF-approximation space. \square

Theorem 3.13 *Let (L, \leq) be a continuous domain, (L, R_L, \mathcal{F}_L) the one constructed in Theorem 3.12. Then $\mathfrak{C}(L, R_L, \mathcal{F}_L) = \{\downarrow x \mid x \in L\}$.*

Proof. By the proof of Theorem 3.12 and Proposition 3.7(1), we know that $\{\downarrow x \mid x \in L\} \subseteq \mathfrak{C}(L, R_L, \mathcal{F}_L)$. Conversely, let $E \in \mathfrak{C}(L, R_L, \mathcal{F}_L)$. Then by Proposition 3.8, there is a directed set $D \subseteq L$ such that $E = \bigcup \{\downarrow d \mid d \in D\}$. Next we prove $E = \downarrow \bigvee D$. Obviously, $E \subseteq \downarrow \bigvee D$. Conversely, if $x \in \downarrow \bigvee D$, then by Lemma 2.2, there is $y \in L$ such that $x \ll y \ll \bigvee D$. So there is $d \in D$ such that $x \ll y \leq d$. Thus $x \in \downarrow d \subseteq E$, and $E = \downarrow \bigvee D$. This shows that $\mathfrak{C}(L, R_L, \mathcal{F}_L) \subseteq \{\downarrow x \mid x \in L\}$ and $\mathfrak{C}(L, R_L, \mathcal{F}_L) = \{\downarrow x \mid x \in L\}$. \square

Theorem 3.14 (Representation Theorem) *Let (L, \leq) be a poset. Then L is a continuous domain iff there exists a CF-approximation space (U, R, \mathcal{F}) such that $(L, \leq) \cong (\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$.*

Proof. \Leftarrow : Follows directly by Theorem 3.11.

\Rightarrow : If L is a continuous domain, then by Theorem 3.12 we know that (L, R_L, \mathcal{F}_L) is a CF-approximation space. Define a map $f : L \rightarrow \mathfrak{C}(L, R_L, \mathcal{F}_L)$ such that for all $x \in L$, $f(x) = \downarrow x$. Then it follows from Theorem 3.13 and the continuity of L that f is an order isomorphism. \square

4 Representations of some special domains

In this section, we add some conditions to CF-approximation spaces, and then discuss representations of some special types of continuous domains.

Lemma 4.1 *Let (L, \leq) be a continuous domain and $B \subseteq L$ a base. If (B, \leq) is a semilattice (resp., sup-semilattice, poset with bottom element, poset with top element, sup-semilattice with bottom element,*

cusl), then (L, \leq) is a continuous semilattice (resp., continuous sup-semilattice, continuous domain with bottom element, continuous domain with top element, continuous lattice, bc-domain).

Proof. (1) Let (B, \leq) be a semilattice. For any $x, y \in L$, set $D = \{a \wedge_B b \mid a \in \downarrow x \cap B, b \in \downarrow y \cap B\}$. It is easy to show that $\downarrow x \cap B$ and $\downarrow y \cap B$ are directed. Therefore D is directed and $\bigvee D$ exists. It is clear that $\bigvee D \leq \bigvee(\downarrow x \cap B) = x$, $\bigvee D \leq \bigvee(\downarrow y \cap B) = y$. If $z \leq x, y$, then for any $t \in \downarrow z \cap B$, we have $t \ll x = \bigvee(\downarrow x \cap B)$, $t \ll y = \bigvee(\downarrow y \cap B)$. Therefore there exist $t_1 \in \downarrow x \cap B$, $t_2 \in \downarrow y \cap B$ such that $t \leq t_1, t_2$. Thus $t \leq t_1 \wedge_B t_2$. Noticing that $t_1 \wedge_B t_2 \in D$ and the arbitrariness of $t \in \downarrow z \cap B$, we have that $z = \bigvee(\downarrow z \cap B) \leq \bigvee D$. This shows that $\bigvee D$ is a greatest lower bound of x, y , namely, $x \wedge y = \bigvee D$. Thus L is a continuous semilattice.

(2) Let (B, \leq) be a sup-semilattice. For any $x, y \in L$, set $D = \{a \vee_B b \mid a \in \downarrow x \cap B, b \in \downarrow y \cap B\}$. Clearly, D is directed and $\bigvee D$ exists. It is obvious that $x, y \leq \bigvee D$. Let $x, y \leq z$. Then for all $a \in \downarrow x \cap B$ and $b \in \downarrow y \cap B$, we have that $a \ll z$, $b \ll z$. By the directedness of $\downarrow z \cap B$, there exists $t \in \downarrow z \cap B$ such that $a, b \leq t$. Therefore $a \vee_B b \leq t$. Noticing that $a \vee_B b \in D$, we have $\bigvee D \leq \bigvee(\downarrow z \cap B) = z$. This shows that $\bigvee D$ is a least upper bound of x, y , namely, $x \vee y = \bigvee D$. Thus L is a continuous sup-semilattice.

(3)/(4) If \perp/\top is a bottom/top element of (B, \leq) , then \perp/\top is also a bottom/top element of L .

(5) Let (B, \leq) be a sup-semilattice with bottom element \perp . Then by (2) and (3), we know that L is a sup-semilattice with bottom element. Since L is a dcpo, L is a complete lattice. Thus L is a continuous lattice.

(6) Let (B, \leq) be a cusl. For any $x, y, z \in L$ which satisfy $x, y \leq z$, we show that for all $a \in \downarrow x \cap B$ and $b \in \downarrow y \cap B$, $a \vee_B b$ exists. By that $x, y \leq z$, we have $a \ll z, b \ll z$. Since B is a base, there exists $c \in \downarrow z \cap B$ such that $a, b \leq c$. That $a \vee_B b$ exists by that (B, \leq) is a cusl. Similar to the proof of (2), we have that L is a cusl. As L is a continuous domain, we see that L is a bc-domain. \square

Theorem 4.2 *Let (U, R, \mathcal{F}) be a CF-approximation space. If $(\{\overline{R}(F) \mid F \in \mathcal{F}\}, \subseteq)$ is a semilattice (resp., sup-semilattice, poset with bottom element, poset with top element, sup-semilattice with bottom element, cusl), then $\mathfrak{C}(U, R, \mathcal{F})$ is a continuous semilattice (resp., continuous sup-semilattice, continuous domain with bottom element, continuous domain with top element, continuous lattice, bc-domain). Conversely, any continuous semilattice (resp., continuous sup-semilattice, continuous domain with bottom element, continuous domain with top element, continuous lattice, bc-domain) L is isomorphic to $(\mathfrak{C}(L, R_L, \mathcal{F}_L), \subseteq)$ of corresponding CF-approximation spaces, respectively.*

Proof. The first half of the theorem follows directly from Theorem 3.11 and Lemma 4.1.

For the second half, let (L, R_L, \mathcal{F}_L) be the one in Theorem 3.12. Define a map $f : L \rightarrow \{\overline{R}(F) \mid F \in \mathcal{F}_L\}$ such that for all $x \in L$, $f(x) = \downarrow x$. Since L is continuous, f is an order isomorphism. Thus $\{\overline{R}(F) \mid F \in \mathcal{F}_L\}$ is a semilattice (resp., sup-semilattice, poset with bottom element, poset with top element, sup-semilattice with bottom element, cusl) whenever L is a continuous semilattice (resp., continuous sup-semilattice, continuous domain with bottom element, continuous domain with top element, continuous lattice, bc-domain). By Theorem 3.13, L is isomorphic to $(\mathfrak{C}(L, R_L, \mathcal{F}_L), \subseteq)$. \square

Next, we consider algebraic cases.

Definition 4.3 *Let (U, R) be a GA-space, $\mathcal{F} \subseteq \mathcal{P}_{fin}(U) \cup \{\emptyset\}$. If R is a preorder, then (U, R, \mathcal{F}) is called a topological CF-approximation space.*

Remark 4.4 *A topological CF-approximation space must be a CF-approximation space. In fact, for all $F \in \mathcal{F}$ and $K \subseteq_{fin} \overline{R}(F)$, taking $G = F$, then by Lemma 2.9 we have $K \subseteq \overline{R}(G)$ and $G \subseteq \overline{R}(F)$. By Definition 3.1, (U, R, \mathcal{F}) is a CF-approximation space.*

Proposition 4.5 *Let (U, R, \mathcal{F}) be a topological CF-approximation space, $E_1, E_2 \in \mathfrak{C}(U, R, \mathcal{F})$. Then $E_1 \ll E_2$ iff there exists $F \in \mathcal{F}$ such that $E_1 \subseteq \overline{R}(F) \subseteq E_2$. Thus $K((\mathfrak{C}(U, R, \mathcal{F}), \subseteq)) = (\{\overline{R}(F) \mid F \in \mathcal{F}\}, \subseteq)$.*

Proof. It follows directly from Lemma 2.9 and Theorem 3.9. \square

Theorem 4.6 *Let (U, R, \mathcal{F}) be a topological CF-approximation space. Then $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an algebraic domain. Conversely, any algebraic domain can be represented by some topological CF-approximation space.*

Proof. The first half of the theorem follows from Proposition 4.5 and Theorem 3.8 that $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an algebraic domain.

For the second half, let (L, \leq) be an algebraic domain. Set $(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$, where $\mathcal{F}_{K(L)} = \{F \subseteq_{fin} K(L) \mid F \text{ has top element}\}$, $R_{K(L)} = \leq$ is a partial order. Thus $(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$ is a topological CF-approximation space. For any $F \in \mathcal{F}_{K(L)}$, let c_F be the top element of F . By Lemma 2.8(5), we know that for all $F \in \mathcal{F}_{K(L)}$, $\overline{R_{K(L)}}(F) = \downarrow c_F \cap K(L)$. Similar to the proof of Theorem 3.13, we have that $\mathfrak{C}(K(L), R_{K(L)}, \mathcal{F}_{K(L)}) = \{\downarrow x \cap K(L) \mid x \in L\}$. Since L is an algebraic domain, we know that $(\{\downarrow x \cap K(L) \mid x \in L\}, \subseteq) \cong (L, \leq)$. The proof is finished. \square

Lemma 4.7 *Let (L, \leq) be an algebraic domain. If $(K(L), \leq)$ is a semilattice, then L is an arithmetic semilattice.*

Proof. For any $x, y \in L$, let $D = \{a \wedge_{K(L)} b \mid a \in \downarrow x \cap K(L), b \in \downarrow y \cap K(L)\}$. By Lemma 4.1, we see that $x \wedge y = \bigvee D$ and L is a semilattice. Next we prove $(K(L), \leq)$ is a subsemilattice of L . If $x, y \in K(L)$, then $x \wedge_{K(L)} y \in D$ and $x \wedge_{K(L)} y$ is the top element of D . So $x \wedge_{K(L)} y = \bigvee D = x \wedge y$. Hence $x \wedge_{K(L)} y = x \wedge y$. This shows that $(K(L), \leq)$ is a subsemilattice of L , and L is an arithmetic semilattice. \square

Theorem 4.8 *Let (U, R, \mathcal{F}) be a topological CF-approximation space. If poset $\{\overline{R}(F) \mid F \in \mathcal{F}\}, \subseteq)$ is a semilattice, then $(\mathfrak{C}(U, R, \mathcal{F}), \subseteq)$ is an arithmetic semilattice. Conversely, any arithmetic semilattice can be represented in this way.*

Proof. The first half of the theorem follows directly from Theorem 4.6 and Lemma 4.7.

For the second half, let L be an arithmetic semilattice and $(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$ be the one in Theorem 4.6. Therefore $(K(L), R_{K(L)}, \mathcal{F}_{K(L)})$ is a topological CF-approximation space. For $F_1, F_2 \in \mathcal{F}_{K(L)}$, then $\overline{R_{K(L)}}(F_1) = \downarrow c_{F_1} \cap K(L)$, $\overline{R_{K(L)}}(F_2) = \downarrow c_{F_2} \cap K(L)$. Since L is an arithmetic semilattice, we know that $c_{F_1} \wedge c_{F_2} = c \in K(L)$. So $\overline{R_{K(L)}}(F_1) \cap \overline{R_{K(L)}}(F_2) = \downarrow c \cap K(L)$. This shows that there exists $\{c\} \in \mathcal{F}_L$ such that $\overline{R_{K(L)}}(F_1) \wedge \overline{R_{K(L)}}(F_2) = \overline{R_{K(L)}}(\{c\})$. This shows that $\{\overline{R_{K(L)}}(F) \mid F \in \mathcal{F}\}$ is a semilattice. By Theorem 4.6, we see that the second half of the theorem holds. \square

5 CF-approximable Relations and Equivalence of Categories

In this section, we define and study CF-approximable relations between CF-approximation spaces and prove that the category of CF-approximation spaces and CF-approximable relations is equivalent to the category of continuous domains and Scott continuous maps.

Definition 5.1 *Let $(U_1, R_1, \mathcal{F}_1)$, $(U_2, R_2, \mathcal{F}_2)$ be CF-approximation spaces, and $\Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ a binary relation. If*

- (1) *for all $F \in \mathcal{F}_1$, there is $G \in \mathcal{F}_2$ such that $F \Theta G$;*
- (2) *for all $F, F' \in \mathcal{F}_1$, $G \in \mathcal{F}_2$, if $F \subseteq \overline{R_1}(F')$, $F \Theta G$, then $F' \Theta G$;*
- (3) *for all $F \in \mathcal{F}_1$, $G, G' \in \mathcal{F}_2$, if $F \Theta G$, $G' \subseteq \overline{R_2}(G)$, then $F \Theta G'$;*
- (4) *for all $F \in \mathcal{F}_1$, $G \in \mathcal{F}_2$, if $F \Theta G$, then there are $F' \in \mathcal{F}_1$, $G' \in \mathcal{F}_2$ such that $F' \subseteq \overline{R_1}(F)$, $G \subseteq \overline{R_2}(G')$ and $F' \Theta G'$; and*
- (5) *for all $F \in \mathcal{F}_1$, $G_1, G_2 \in \mathcal{F}_2$, if $F \Theta G_1$ and $F \Theta G_2$, then there is $G_3 \in \mathcal{F}_2$ such that $G_1 \cup G_2 \subseteq \overline{R_2}(G_3)$ and $F \Theta G_3$,*

then Θ is called a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$.

Proposition 5.2 *Let Θ be a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. Then for all $F \in \mathcal{F}_1, G \in \mathcal{F}_2$, the following statements are equivalent:*

- (1) $F \Theta G$;
- (2) There exists $F' \in \mathcal{F}_1$ such that $F' \subseteq \overline{R_1}(F)$ and $F' \Theta G$;
- (3) There exists $G' \in \mathcal{F}_2$ such that $F \Theta G'$ and $G \subseteq \overline{R_2}(G')$;
- (4) There exist $F' \in \mathcal{F}_1$ and $G' \in \mathcal{F}_2$ such that $F' \subseteq \overline{R_1}(F)$, $G \subseteq \overline{R_2}(G')$ and $F' \Theta G'$.

Proof. Follows directly from Definition 5.1(2)-(4). \square

Let Θ be a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. For all $F \in \mathcal{F}_1$, set $\tilde{\Theta}(F) = \bigcup\{\overline{R_2}(G) \mid F \Theta G \text{ and } G \in \mathcal{F}_2\}$. Define a map $f_\Theta : \mathfrak{C}(U_1, R_1, \mathcal{F}_1) \rightarrow \mathcal{P}(U_2)$ such that for all $E \in \mathfrak{C}(U_1, R_1, \mathcal{F}_1)$, $f_\Theta(E) = \bigcup\{\tilde{\Theta}(F) \mid F \subseteq E \text{ and } F \in \mathcal{F}_1\}$.

Proposition 5.3 *Let Θ be a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$, $F \in \mathcal{F}_1$, $E \in \mathfrak{C}(U_1, R_1, \mathcal{F}_1)$. Then the following hold:*

- (1) $\{\overline{R_2}(G) \mid F \Theta G \text{ and } G \in \mathcal{F}_2\}$ is directed;
- (2) $\tilde{\Theta}(F) \in \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$;
- (3) For any $F \in \mathcal{F}_1$, $f_\Theta(\overline{R_1}(F)) = \tilde{\Theta}(F)$;
- (4) $\{\tilde{\Theta}(F) \mid F \subseteq E, F \in \mathcal{F}_1\}$ is directed and $f_\Theta(E) \in \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$.

Proof. (1) By Definition 5.1(1), we know that $\{\overline{R_2}(G) \mid F \Theta G, G \in \mathcal{F}_2\}$ is not empty. By Definition 5.1(5) and Lemma 3.2, we know that $\{\overline{R_2}(G) \mid F \Theta G, G \in \mathcal{F}_2\}$ is directed.

(2) Follows directly from (1) and Proposition 3.7(1).

(3) It is easy to check that

$$\begin{aligned} f_\Theta(\overline{R_1}(F)) &= \bigcup\{\tilde{\Theta}(F') \mid F' \subseteq \overline{R_1}(F), F' \in \mathcal{F}_1\} \\ &= \bigcup\{\overline{R_2}(G) \mid F' \in \mathcal{F}_1, G \in \mathcal{F}_2, F' \subseteq \overline{R_1}(F) \text{ and } F' \Theta G\} \quad (\text{by the definition of } \tilde{\Theta}(F')) \\ &= \bigcup\{\overline{R_2}(G) \mid G \in \mathcal{F}_2, F \Theta G\} \quad (\text{by Proposition 5.2}) \\ &= \tilde{\Theta}(F) \quad (\text{by the definition of } \tilde{\Theta}(F)). \end{aligned}$$

(4) Firstly, we show that $\{\tilde{\Theta}(F) \mid F \subseteq E, F \in \mathcal{F}_1\}$ is directed. Let $F_1, F_2 \in \mathcal{F}_1$. If $F_1, F_2 \subseteq E$, then by Proposition 3.8(4), there exists $F_3 \in \mathcal{F}_1$ such that $F_1 \cup F_2 \subseteq \overline{R_1}(F_3) \subseteq E$. And by Definition 5.1(2), it is easy to deduce that $\tilde{\Theta}(F_1), \tilde{\Theta}(F_2) \subseteq \tilde{\Theta}(F_3)$. This shows that the family $\{\tilde{\Theta}(F) \mid F \subseteq E, F \in \mathcal{F}_1\}$ is directed. Noticing that $f_\Theta(E) = \bigcup\{\tilde{\Theta}(F) \mid F \subseteq E, F \in \mathcal{F}_1\}$, by (2) and Proposition 3.7(3), we have that $f_\Theta(E) \in \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$. \square

Proposition 5.3 shows that f_Θ can be seen as a map from $\mathfrak{C}(U_1, R_1, \mathcal{F}_1)$ to $\mathfrak{C}(U_2, R_2, \mathcal{F}_2)$.

Proposition 5.4 *Let Θ be a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$, $F \in \mathcal{F}_1$, $G \in \mathcal{F}_2$. Then $G \subseteq \tilde{\Theta}(F) \Leftrightarrow F \Theta G$.*

Proof. It is easy to check that

$$\begin{aligned}
G \subseteq \tilde{\Theta}(F) &\Leftrightarrow G \subseteq \bigcup \{ \overline{R_2}(G') \mid F \Theta G', G' \in \mathcal{F}_2 \} \\
&\Leftrightarrow \exists G' \in \mathcal{F}_2 \text{ s.t. } F \Theta G', G \subseteq \overline{R_2}(G') \quad (\text{by Proposition 5.3(1) and the finiteness of } G) \\
&\Leftrightarrow F \Theta G \quad (\text{by Proposition 5.2}). \quad \square
\end{aligned}$$

Theorem 5.5 *Let Θ be a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. Then f_Θ is a Scott continuous map from $\mathfrak{C}(U_1, R_1, \mathcal{F}_1)$ to $\mathfrak{C}(U_2, R_2, \mathcal{F}_2)$.*

Proof. It follows from the definition of f_Θ that f_Θ is order preserving. In order to prove f_Θ is Scott continuous, by Proposition 3.7(3), it suffices to show that for any directed family $\{E_i\}_{i \in I} \subseteq \mathfrak{C}(U_1, R_1, \mathcal{F}_1)$, we have $f_\Theta(\bigcup_{i \in I} E_i) = \bigcup_{i \in I} f_\Theta(E_i)$. In fact,

$$\begin{aligned}
f_\Theta(\bigcup_{i \in I} E_i) &= \bigcup \{ \tilde{\Theta}(F) \mid F \subseteq \bigcup_{i \in I} E_i, F \in \mathcal{F}_1 \} \\
&= \bigcup_{i \in I} (\bigcup \{ \tilde{\Theta}(F) \mid F \subseteq E_i, F \in \mathcal{F}_1 \}) \\
&\quad (\text{by the finiteness of } F \text{ and the directedness of } \{E_i\}_{i \in I}) \\
&= \bigcup_{i \in I} f_\Theta(E_i) \quad (\text{by the definition of } f_\Theta(E_i)). \quad \square
\end{aligned}$$

Theorem 5.5 shows that a CF-approximable relation between CF-approximation spaces can induce a Scott continuous map between continuous domains. Conversely, a Scott continuous map between relative continuous domains can also induce a CF-approximable relation between CF-approximation spaces as follows.

Theorem 5.6 *Let $f : \mathfrak{C}(U_1, R_1, \mathcal{F}_1) \rightarrow \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$ be a Scott continuous map between CF-approximation spaces $(U_1, R_1, \mathcal{F}_1)$ and $(U_2, R_2, \mathcal{F}_2)$. Define $\Theta_f \subseteq \mathcal{F}_1 \times \mathcal{F}_2$ such that*

$$\forall F \in \mathcal{F}_1, G \in \mathcal{F}_2, F \Theta_f G \Leftrightarrow G \subseteq f(\overline{R_1}(F)).$$

Then Θ_f is a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$.

Proof. It follows from $f(\overline{R_1}(F)) \in \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$ and Definition 3.3 that Θ_f satisfies Definition 5.1(1).

To check that Θ_f satisfies Definition 5.1(2), let $F, F' \in \mathcal{F}_1, G \in \mathcal{F}_2$. Then

$$\begin{aligned}
&F \subseteq \overline{R_1}(F'), F \Theta_f G \\
&\Rightarrow F \subseteq \overline{R_1}(F'), G \subseteq f(\overline{R_1}(F)) \quad (\text{by the definition of } \Theta_f) \\
&\Rightarrow G \subseteq f(\overline{R_1}(F)) \subseteq f(\overline{R_1}(F')) \quad (\text{by Lemma 3.2 and the order preservation of } f) \\
&\Rightarrow G \subseteq f(\overline{R_1}(F')) \Leftrightarrow F' \Theta_f G.
\end{aligned}$$

To check that Θ_f satisfies Definition 5.1(3), let $F \in \mathcal{F}_1, G, G' \in \mathcal{F}_2$. Then

$$\begin{aligned}
&F \Theta_f G, G' \subseteq \overline{R_2}(G) \\
&\Rightarrow G \subseteq f(\overline{R_1}(F)), G' \subseteq \overline{R_2}(G) \\
&\Rightarrow G' \subseteq \overline{R_2}(G) \subseteq f(\overline{R_1}(F)) \quad (\text{by Proposition 3.7(2) and } f(\overline{R_1}(F)) \in \mathfrak{C}(U_2, R_2, \mathcal{F}_2)) \\
&\Rightarrow G' \subseteq f(\overline{R_1}(F)) \Leftrightarrow F \Theta_f G'.
\end{aligned}$$

To check that Θ_f satisfies Definition 5.1(4), let $F \in \mathcal{F}_1$, $G \in \mathcal{F}_2$. If $F \Theta_f G$, then $G \subseteq f(\overline{R_1}(F))$. By Proposition 3.7, 3.8(2) and the Scott continuity of f , we know that

$$(*) \quad f(\overline{R_1}(F)) = \bigcup \{f(\overline{R_1}(F')) \mid F' \subseteq \overline{R_1}(F), F' \in \mathcal{F}_1\}.$$

Therefore we have that

$$\begin{aligned} G \subseteq f(\overline{R_1}(F)) &\Rightarrow \exists G' \in \mathcal{F}_2, \text{ s.t. } G \subseteq \overline{R_2}(G') \text{ and } G' \subseteq f(\overline{R_1}(F)) \quad (\text{by Definition 3.3}) \\ &\Rightarrow \exists F' \in \mathcal{F}_1, G' \in \mathcal{F}_2, \text{ s.t. } G \subseteq \overline{R_2}(G'), F' \subseteq \overline{R_1}(F) \text{ and } G' \subseteq f(\overline{R_1}(F')) \\ &\quad (\text{by equation } (*) \text{ and the finiteness of } G') \\ &\Rightarrow \exists F' \in \mathcal{F}_1, G' \in \mathcal{F}_2, \text{ s.t. } G \subseteq \overline{R_2}(G'), F' \subseteq \overline{R_1}(F) \text{ and } F' \Theta_f G' \\ &\quad (\text{by the definition of } \Theta_f). \end{aligned}$$

To check that Θ_f satisfies Definition 5.1(5), let $F \in \mathcal{F}_1$, $G_1, G_2 \in \mathcal{F}_2$. If $F \Theta_f G_1$ and $F \Theta_f G_2$, then $G_1 \cup G_2 \subseteq f(\overline{R_1}(F))$. By Definition 3.3 and $f(\overline{R_1}(F)) \in \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$, there exists $G_3 \in \mathcal{F}_2$ such that $G_1 \cup G_2 \subseteq \overline{R_2}(G_3) \subseteq f(\overline{R_1}(F))$ and $G_3 \subseteq f(\overline{R_1}(F))$. So $F \Theta_f G_3$, showing that Θ_f satisfies Definition 5.1(5). \square

Theorem 5.7 *Let $f : \mathfrak{C}(U_1, R_1, \mathcal{F}_1) \rightarrow \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$ be a Scott continuous map between CF-approximation spaces $(U_1, R_1, \mathcal{F}_1)$ and $(U_2, R_2, \mathcal{F}_2)$, Θ a CF-approximable relation from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. Then $\Theta_{f_\Theta} = \Theta$ and $f_{\Theta_f} = f$.*

Proof. Let $F \in \mathcal{F}_1, G \in \mathcal{F}_2$. Then by Propositions 5.3(3) and 5.4, we have

$$(F, G) \in \Theta_{f_\Theta} \Leftrightarrow G \subseteq f_\Theta(\overline{R_1}(F)) = \widetilde{\Theta}(F) \Leftrightarrow (F, G) \in \Theta,$$

showing that $\Theta_{f_\Theta} = \Theta$.

For any $E \in \mathfrak{C}(U_1, R_1, \mathcal{F}_1)$, we have that

$$\begin{aligned} f_{\Theta_f}(E) &= \bigcup \{\widetilde{\Theta}_f(F) \mid F \subseteq E \text{ and } F \in \mathcal{F}_1\} \\ &= \bigcup \{\overline{R_2}(G) \mid F \in \mathcal{F}_1, G \in \mathcal{F}_2, F \subseteq E \text{ and } F \Theta_f G\} \\ &= \bigcup \{\overline{R_2}(G) \mid F \in \mathcal{F}_1, G \in \mathcal{F}_2, F \subseteq E \text{ and } G \subseteq f(\overline{R_1}(F))\} \\ &= \bigcup \{f(\overline{R_1}(F)) \mid F \in \mathcal{F}_1, F \subseteq E\} \quad (\text{by Proposition 3.8(2)}) \\ &= f(\bigcup \{\overline{R_1}(F) \mid F \in \mathcal{F}_1, F \subseteq E\}) \quad (\text{by Scott continuity of } f) \\ &= f(E). \end{aligned}$$

This shows that $f_{\Theta_f} = f$. \square

Given a CF-approximation space (U, R, \mathcal{F}) , define the identity on (U, R, \mathcal{F}) to be a binary relation $\text{Id}_{(U, R, \mathcal{F})} \subseteq \mathcal{F} \times \mathcal{F}$ such that for all $F, G \in \mathcal{F}$, $(F, G) \in \text{Id}_{(U, R, \mathcal{F})} \Leftrightarrow G \subseteq \overline{R}(F)$.

Let $(U_1, R_1, \mathcal{F}_1)$, $(U_2, R_2, \mathcal{F}_2)$, $(U_3, R_3, \mathcal{F}_3)$ be CF-approximation spaces, $\Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_2$, $\Upsilon \subseteq \mathcal{F}_2 \times \mathcal{F}_3$ be CF-approximable relations. Define $\Upsilon \circ \Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_3$, the composition of Υ and Θ by that for any $F_1 \in \mathcal{F}_1, F_3 \in \mathcal{F}_3$, $(F_1, F_3) \in \Upsilon \circ \Theta$ iff there exists $F_2 \in \mathcal{F}_2$ satisfying $(F_1, F_2) \in \Theta$ and $(F_2, F_3) \in \Upsilon$.

It is a routine work to check that $\text{Id}_{(U, R, \mathcal{F})}$ is a CF-approximable relation from (U, R, \mathcal{F}) to itself. Thus, CF-approximation spaces as objects and CF-approximable relations as morphisms with the identities and compositions defined above, form a category, and denoted by **CF-GA**.

Let **CDOM** be the category of continuous domains and Scott continuous maps. We next show that categories **CF-GA** and **CDOM** are equivalent.

Lemma 5.8 ([1]) *Let \mathcal{C}, \mathcal{D} be two categories. If there is a functor $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ such that*

(1) Φ is full, namely, for all $A, B \in \text{ob}(\mathcal{C})$, $g \in \text{Mor}_{\mathcal{D}}(\Phi(A), \Phi(B))$, there is $f \in \text{Mor}_{\mathcal{C}}(A, B)$ such that $\Phi(f) = g$;

(2) Φ is faithful, namely, for all $A, B \in \text{ob}(\mathcal{C})$, $f, g \in \text{Mor}_{\mathcal{C}}(A, B)$, if $f \neq g$, then $\Phi(f) \neq \Phi(g)$;

(3) for all $B \in \text{ob}(\mathcal{D})$, there is $A \in \text{ob}(\mathcal{C})$ such that $\Phi(A) \cong B$,

then \mathcal{C} and \mathcal{D} are equivalent.

Theorem 5.9 *The categories **CF-GA** and **CDOM** are equivalent.*

Proof. Define $\Psi : \mathbf{CF-GA} \rightarrow \mathbf{CDOM}$ such that for all $(U, R, \mathcal{F}) \in \text{ob}(\mathbf{CF-GA})$, $\Psi((U, R, \mathcal{F})) = (\mathfrak{C}(U, R, \mathcal{F}), \subseteq) \in \text{ob}(\mathbf{CDOM})$; for all $\Theta \in \text{Mor}(\mathbf{CF-GA})$, $\Psi(\Theta) = f_{\Theta} \in \text{Mor}(\mathbf{CDOM})$.

Give a CF-approximation space (U, R, \mathcal{F}) , for any $E \in \mathfrak{C}(U, R, \mathcal{F})$, we have

$$\begin{aligned} \Psi(\text{Id}_{(U, R, \mathcal{F})})(E) &= f_{\text{Id}_{(U, R, \mathcal{F})}}(E) \\ &= \bigcup \{ \overline{R}(G) \mid F, G \in \mathcal{F}, F \subseteq E \text{ and } (F, G) \in \text{Id}_{(U, R, \mathcal{F})} \} \\ &= \bigcup \{ \overline{R}(G) \mid F, G \in \mathcal{F}, F \subseteq E \text{ and } G \subseteq \overline{R}(F) \} \\ &= \bigcup \{ \overline{R}(F) \mid F \in \mathcal{F}, F \subseteq E \} \quad (\text{by Proposition 3.7(1) and 3.8(2)}) \\ &= E \quad (\text{by Proposition 3.8(2)}) \\ &= \text{id}_{\mathfrak{C}(U, R, \mathcal{F})}(E). \end{aligned}$$

This shows that $\Psi(\text{Id}_{(U, R, \mathcal{F})}) = \text{id}_{\mathfrak{C}(U, R, \mathcal{F})}$.

Let $\Theta \subseteq \mathcal{F}_1 \times \mathcal{F}_2$, $\Upsilon \subseteq \mathcal{F}_2 \times \mathcal{F}_3$ be CF-approximable relations. Then for any $E \in \mathfrak{C}(U, R, \mathcal{F})$, we have

$$\begin{aligned} \Psi(\Upsilon) \circ \Psi(\Theta)(E) &= f_{\Upsilon}(f_{\Theta}(E)) \\ &= \bigcup \{ \widetilde{\Upsilon}(F) \mid F \subseteq f_{\Theta}(E), F \in \mathcal{F}_2 \} \\ &= \bigcup \{ \overline{R}_3(G) \mid F \subseteq f_{\Theta}(E), F \in \mathcal{F}_2, G \in \mathcal{F}_3 \text{ and } F \Upsilon G \} \quad (\text{by the definition of } \widetilde{\Upsilon}) \\ &= \bigcup \{ \overline{R}_3(G) \mid F_1 \in \mathcal{F}_1, F_1 \subseteq E, G_1 \in \mathcal{F}_2, F_1 \Theta G_1, F \subseteq \overline{R}_2(G_1), F \in \mathcal{F}_2, G \in \mathcal{F}_3 \text{ and } F \Upsilon G \} \\ &\quad (\text{by the definition of } f_{\Theta}(E), \text{ Proposition 5.3(1), Theorem 5.5 and finiteness of members in } \mathcal{F}_2) \\ &= \bigcup \{ \overline{R}_3(G) \mid F_1 \in \mathcal{F}_1, F_1 \subseteq E, G_1 \in \mathcal{F}_2, F_1 \Theta G_1, G \in \mathcal{F}_3 \text{ and } G_1 \Upsilon G \} \\ &\quad (\text{by } F \subseteq \overline{R}_2(G_1), F \Upsilon G, \text{ and Definition 5.1(2)}) \\ &= \bigcup \{ \overline{R}_3(G) \mid F_1 \in \mathcal{F}_1, F_1 \subseteq E, G \in \mathcal{F}_3 \text{ and } (F_1, G) \in \Upsilon \circ \Theta \} \quad (\text{by } F_1 \Theta G_1 \text{ and } G_1 \Upsilon G) \\ &= \bigcup \{ \widetilde{\Upsilon \circ \Theta}(F_1) \mid F_1 \in \mathcal{F}_1, F_1 \subseteq E \} \quad (\text{by the definition of } \widetilde{\Upsilon \circ \Theta}(F_1)) \\ &= f_{\Upsilon \circ \Theta}(E) = \Psi(\Upsilon \circ \Theta)(E). \end{aligned}$$

This shows that $\Psi(\Upsilon) \circ \Psi(\Theta) = \Psi(\Upsilon \circ \Theta)$, and thus Ψ is a functor.

To show that **CF-GA** is equivalent to **CDOM**, it suffices to check that Ψ satisfies the three conditions in Lemma 5.8.

Let $(U_1, R_1, \mathcal{F}_1), (U_2, R_2, \mathcal{F}_2)$ be CF-approximation spaces, Θ_1, Θ_2 be CF-approximable relations from $(U_1, R_1, \mathcal{F}_1)$ to $(U_2, R_2, \mathcal{F}_2)$. If $\Theta_1 \neq \Theta_2$, then by Theorem 3.14 we know that $\Theta_1 = \Theta_{f_{\Theta_1}} \neq \Theta_{f_{\Theta_2}} = \Theta_2$. Thus $f_{\Theta_1} \neq f_{\Theta_2}$, showing that Ψ is faithful.

Let $f : \mathfrak{C}(U_1, R_1, \mathcal{F}_1) \rightarrow \mathfrak{C}(U_2, R_2, \mathcal{F}_2)$ be a Scott continuous map, by Theorem 3.14, there is $\Theta_f \in \text{Mor}(\mathbf{CF-GA})$ such that $\Psi(\Theta_f) = f_{\Theta_f} = f$, showing that Ψ is full.

It is clear by Theorem 3.14 that Ψ satisfies the condition (3) in Lemma 5.8. \square

Similarly, we can also establish categorical equivalences between the category of algebraic domains with Scott continuous maps as morphisms and the category of topological CF-approximation spaces with CF-approximable relations as morphisms. We leave the details to the interested readers.

6 Conclusions

This paper generalizes abstract bases to CF-approximation spaces and generalizes the family of round ideals of an abstract basis to the family of CF-closed sets of a CF-approximation space. Thus a representation method of various continuous domains including continuous semilattices, continuous sup-semilattices, continuous domains with bottom, continuous domains with top, continuous lattices, bc-domains, algebraic domains and arithmetic semilattices in the framework of rough set theory is obtained. CF-approximable relations between CF-approximation spaces are defined, and categorical equivalence between categories **CF-GA** of CF-approximation spaces and continuous domains and **CDOM** of continuous domains and Scott continuous maps is established. This work strengthens the links among rough set theory, domain theory and topology, and widens the scope of application of rough set theory and domain theory.

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