The Sheaf Representation of Residuated Lattices

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Abstract

The residuated lattices form one of the most important algebras of fuzzy logics and have been heavily studied by people from various different points of view. Sheaf presentations provide a topological approach to many algebraic structures. In this paper, we study the topological properties of prime spectrum of residuated lattices, and then construct a sheaf space to obtain a sheaf representation for each residuated lattice.

Keywords: Residuated lattice, sheaf representation, prime spectrum.

1 Introduction

Residuated lattices, introduced by Ward and Dilworth [20], constitute the semantics of Höhle's Monoidal Logic [14]. Such algebras provide the fundamental framework for algebras of logics. Many familiar algebras, such as Boolean algebras, MV-algebras, BL-algebras, MTL-algebras, NM algebras (R_0-algebras) and Heyting algebras, are special types of residuated lattices.

In dealing with certain type of problems, sheaf representations of algebras often provide powerful tools as they convert the study of algebras to the study of stalks, a topological structure. Thus, in the past decades, sheaf spaces [2,17,18] have been constructed for various types of algebras to obtain their corresponding sheaf representations.

In the case of algebras for fuzzy logics, Ghilardi and Zawadowski constructed the Grothendieck-type duality and got the sheaf representation for Heyting algebras [10]. Many scholars investigated the sheaf representations of MV-algebras [3,5,6,7,8,9]. Here we give an outline of their differences. In [5], Dubuc and

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Poveda use $\mathrm{Spec}(L)$ (which is endowed with the co-Zariski topology) as the base space and MV-chains as the stalks. In [7], Filipoiu and Georgescu used $\mathrm{Max}(L)$ (which is endowed with the Zariski topology) as the base space and local MV-algebras as the stalks. Di Nola, Esposito and Gerla [3] improved the methods of [7], by choosing the stalks from given classes of local MV-algebras. Ferraioli and Lettieri [6], combining the techniques in [5] and [7], got two types of sheaf representation of MV-algebras. In [8] and [9], Gehrke, Gool and Marra provided a general framework for previously known results on sheaf representations of MV-algebras as [5] and [7], through the lens of Stone-Priestley duality, using canonical extensions as an essential tool. In terms of the sheaf representations of BL-algebras, Di Nola and Leuştean, who adopted $\mathrm{Spec}(L)(\mathrm{Max}(L))$ as the base space and BL-algebras (local BL-algebras) as the stalks, obtained the sheaf representation and the compact representation of BL-algebras [3,16].

In [8], Gehrke and Gool dealt with sheaf representations of a $\mathcal{V}$-algebra. In their definition Def 3.1 [8], they required that the $\mathcal{V}$-algebra A is isomorphic to FY, where F is a sheaf, and FY is the algebra of global sections of F. In this paper, we loosen the isomorphism condition in [8] and the requirement on the stalks in [6] and further extend above results to the more general structures; namely, define the sheaf spaces of residuated lattices and obtain the sheaf representation of residuated lattices.

2 Preliminaries

In this section, we recall some basic notions and results to be used in the sequel.

Definition 2.1 ([20]) A residuated lattice is an algebra $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ satisfying the following conditions:

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \otimes, 1)$ is a commutative monoid with identity 1;
(3) for any $x, y, z \in L$, $x \otimes y \leq z$ iff $x \leq y \rightarrow z$.

In the following, we shall use the shorthand L to denote $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$.

Definition 2.2 ([13,19]) A nonempty subset F of a residuated lattice L is called a filter if

(1) $x \in F$ and $x \leq y$ imply $y \in F$;
(2) $x \in F$ and $y \in F$ imply $x \otimes y \in F$.

Theorem 2.3 *([13,19]) A nonempty subset F of a residuated lattice L is a filter if*

(1) $1 \in F$;
(2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

Remark 2.4 A filter is *proper* if $F \neq L$. We will use $\mathcal{F}(L)$ to denote the set of all filters of a residuated lattice L. Note that $\{1\}$ and L are filters. For any $X \subseteq L$, the filter of L generated by X (the smallest filter containing X) will be denoted by $<X>$. In particular, the filter generated by $\{a\}$ will be denoted by $<a>$. To each filter F, we can associate a congruence relation $\equiv_F$ on L given by

$$x \equiv_F y \text{ iff } ((x \rightarrow y) \otimes (y \rightarrow x)) \in F.$$

Let L/F denote the set of the congruence classes of $\equiv_F$, i.e., $L/F = \{x/F | x \in L\}$, where $x/F := \{y \in L | y \equiv_F x\}$. Define operations on L/F as follows:

$$x/F \sqcap y/F = (x \wedge y)/F, \ x/F \sqcup y/F = (x \vee y)/F,$$
$$x/F \odot y/F = (x \otimes y)/F, \ x/F \rightharpoonup y/F = (x \rightarrow y)/F.$$

Remark 2.5 ([11,12]) It is easy to show that if $\{F_i : i \in I\} \subseteq \mathcal{F}(L)$ is a directed family ($\forall i_1, i_2 \in I, \exists i_s \in I$ such that $F_{i_1} \subseteq F_{i_s}$ and $F_{i_2} \subseteq F_{i_s}$), then $\bigcup_{i \in I} F_i \in \mathcal{F}(L)$. Thus, for any $x \in L$, $<x> \ll <x>$ holds in the complete lattice $(\mathcal{F}(L), \subseteq)$ (see [11,12] for the definition of the way below relation $\ll$). It's not clear

whether $F \in \mathcal{F}(L)$ and $F \ll F$ implies $F = <x>$ for some $x \in L$.

Lemma 2.6 *([1]) Let F be a filter of a residuated lattice L. Then $(L/F, \sqcap, \sqcup, \odot, \rightharpoonup, 0/F, 1/F)$ is a residuated lattice.*

Remark 2.7 ([1]) The following properties hold on any residuated lattice:

(1) $x \vee (y \otimes z) \geq (x \vee y) \otimes (x \vee z)$;

(2) $<x> \cap <y> = <x \vee y>$.

Definition 2.8 ([1]) A proper filter P of a residuated lattice L is called a prime filter if for any $x, y \in L$, $x \vee y \in P$ implies $x \in P$ or $y \in P$.

The set of all prime filters of a residuated lattice L is called the *prime spectrum* of L and is denoted by $\mathrm{Spec}(L)$.

Lemma 2.9 *Let P be a proper filter of a residuated lattice L. Then P is a prime filter iff $F_1 \cap F_2 \subseteq P$ implies $F_1 \subseteq P$ or $F_2 \subseteq P$ for any $F_1, F_2 \in \mathcal{F}(L)$.*

Proof. Assume that $F_1 \cap F_2 \subseteq P$ with $F_1 \not\subseteq P$ and $F_2 \not\subseteq P$. Then there exist $x \in F_1, y \in F_2$ such that $x \notin P$, $y \notin P$. Thus $x \vee y \in F_1 \cap F_2$ and $x \vee y \notin P$, since P is a prime filter. This shows that $F_1 \cap F_2 \not\subseteq P$, a contradiction. Conversely, assume that $x \vee y \in P$ with $x \notin P$ and $y \notin P$. Thus $<x> \not\subseteq P$ and $<y> \not\subseteq P$. Therefore $<x> \cap <y> \not\subseteq P$. We have $<x \vee y> \not\subseteq P$, that is, $x \vee y \notin P$, again a contradiction.

Next, for any $X \subseteq L$, we will write $D(X) = \{P \in \mathrm{Spec}(L) | X \not\subseteq P\}$. For any $a \in L$, $D(\{a\})$ shall be denoted simply by $D(a)$. $\square$

Lemma 2.10 *Let L be a residuated lattice. Then*

(1) $X \subseteq Y \subseteq L$ implies $D(X) \subseteq D(Y)$;

(2) $D(X) = D(<X>)$.

Proof. (1) is trivial.

(2) Since $X \subseteq <X>$, from (1), we have that $D(X) \subseteq D(<X>)$. Conversely, suppose $P \in D(<X>)$, then $<X> \not\subseteq P$. It follows, by the definition of $<X>$, that $X \not\subseteq P$. That is, $P \in D(X)$. Thus, we have $D(X) = \mathrm{D}(<X>)$. $\square$

We now recall some basic notions about topology to be used later. For more about these, we refer to [15].

A topological space is a pair (X, τ), where X is a nonempty set and τ is a family of subsets of X, called the topology, such that (i) $\emptyset, X \in \tau$, (ii) a finite intersection of members of τ is in τ and (iii) an arbitrary union of members of τ is in τ.

The members of τ are called *open sets* of X and the elements of X are called *points*. A *neighbourhood of a point x* in a topological space X is a subset $W \subseteq X$ such that there exists an open set U of X satisfying $x \in U \subseteq W$. A set U is open iff U is the neighbourhood of every $x \in U$. A *base $\mathcal{B}$* for a topology τ is a collection of open sets in τ such that every open set in τ is a union of some members of $\mathcal{B}$.

Lemma 2.11 *A collection $\mathcal{B}$ of subsets of set X is the base for some topology iff $X = \bigcup\{V : V \in \mathcal{B}\}$ and if $V_1, V_2 \in \mathcal{B}, x \in V_1 \cap V_2$, then there exists $V \in \mathcal{B}$ such that $x \in V \subseteq V_1 \cap V_2$.*

A function $f : X \longrightarrow Y$ from a topological space (X, τ) to a topological space (Y, σ) is *continuous at a point $x \in X$* if for any neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$. The function is called *continuous* if it is continuous everywhere. For any function $f : X \longrightarrow Y$ between two topological spaces, f is continuous iff for any open set W of Y, $f^{-1}(W)$ is open in X iff for any open set V in a base $\mathcal{B}$ of Y, $f^{-1}(V)$ is open in X. A function $f : X \longrightarrow Y$ between two topological spaces X

and Y is an *open function* if for any open set U of X, $f(U)$ is an open set of Y. A function $f : X \longrightarrow Y$ between two topological spaces X and Y is an open function iff for any open set W in a base of X, $f(W)$ is open in Y. A bijective function $f : X \longrightarrow Y$ between two topological spaces is a *homeomorphism* if both f and f^{-1} are continuous. A bijective function $f : X \longrightarrow Y$ between two topological spaces is a homeomorphism iff f is continuous and open.

Theorem 2.12 *For any residuated lattice L, the family $\{D(X)|X \subseteq L\}$ is a topology on $Spec(L)$, which we call the* Stone topology *on L.*

Proof. We complete the proof by verifying each of the following.

(1) $D(L) = \mathrm{Spec}(L)$ and $D(1) = \emptyset$.

(2) For any $X \subseteq L$ and $Y \subseteq L$, $D(X) \bigcap D(Y) = D(<\mathrm{X}>\cap<\mathrm{Y}>)$.

(3) For any family $\{X_i|i \in I\}$ of subsets of L, $D(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} D(X_i)$.

For any $P \in \mathrm{Spec}(L), L \nsubseteq P$. Thus $D(L) = \mathrm{Spec}(L)$. For any $P \in \mathrm{Spec}(L), \{1\} \subseteq P$. Hence $P \notin D(1)$. Therefore $D(1) = \emptyset$. Thus (1) holds.

Since $<\mathrm{X}>\cap<\mathrm{Y}>\subseteq<\mathrm{X}>,<\mathrm{Y}>$, by Lemma 2.10, we have $D(<\mathrm{X}>\cap<\mathrm{Y}>) \subseteq D(<\mathrm{X}>) \bigcap D(<\mathrm{Y}>)$ $= D(X) \bigcap D(Y)$. Conversely, suppose that $P \in D(X) \bigcap D(Y)$, then $X \nsubseteq P$ and $Y \nsubseteq P$. Hence $<\mathrm{X}>\nsubseteq P$ and $<\mathrm{Y}>\nsubseteq P$. By Lemma 2.9, we have $<\mathrm{X}>\cap<\mathrm{Y}>\nsubseteq P$. This shows that $P \in D(<\mathrm{X}>\cap<\mathrm{Y}>)$. Therefore $D(X) \bigcap D(Y) = D(<\mathrm{X}>) \bigcap D(<\mathrm{Y}>)\subseteq D(<\mathrm{X}>\cap<\mathrm{Y}>)$. Hence (2) holds.

Lastly, we verify (3). Suppose that $P \in D(\bigcup_{i \in I} X_i)$, then there exists $i \in I$ such that $X_i \nsubseteq P$. Thus we have $P \in D(X_i) \subseteq \bigcup_{i \in I} D(X_i)$. Hence $D(\bigcup_{i \in I} X_i) \subseteq \bigcup_{i \in I} D(X_i)$. The reverse inclusion holds by Lemma 2.10 (1). $\qquad\square$

Remark 2.13 By Lemma 2.10 (2) and Theorem 2.12, we know that the open sets in the spectrum $\mathrm{Spec}(L)$ are exactly the subsets in $\{D(F) : F \in \mathcal{F}(L)\}$.

Theorem 2.14 *For any residuated lattice L, the family $\{D(a)\}_{a \in L}$ is a base for the* Stone topology *on* $\mathrm{Spec}(L)$.

Proof. Suppose that $X \subseteq L$ and $D(X)$ is an arbitrary open set of $\mathrm{Spec}(L)$, then $D(X) = D(\bigcup_{a \in X}\{a\}) = \bigcup_{a \in X} D(a)$. Hence every open set U of $\mathrm{Spec}(L)$ is the union of a subset of $\{D(a)\}_{a \in U}$. $\qquad\square$

Proposition 2.15 *For any $P \in \mathrm{Spec}(L)$, $O(P)$ is a proper filter of a residuated lattice L satisfying $O(P) \subseteq P$, where $O(P) = \{x \in L|a \lor x = 1 \text{ for some } a \in L - P\}$.*

Proof. Since $1 \notin L - P$, it follows immediately that $0 \notin O(P)$. If $x \in O(P)$ and $x \leq y$, then there exists $a \in L - P$ such that $a \lor x = 1$. Hence $1 = x \lor a \leq y \lor a$. Therefore $y \lor a = 1$, showing that $y \in O(P)$. Next, if $x, y \in O(P)$, then there exist $a, b \in L - P$ such that $a \lor x = 1$ and $b \lor y = 1$. So $a \lor b \in L - P$, because P is a prime filter of L. Thus $(a \lor b) \lor (x \otimes y) \geq (a \lor b \lor x) \otimes (a \lor b \lor y) = 1 \otimes 1 = 1$. Therefore $(a \lor b) \lor (x \otimes y) = 1$. This shows that $x \otimes y \in O(P)$. For any $x \in O(P)$, there exists $a \in L - P$ such that $a \lor x = 1 \in P$. Thus $x \in P$. $\qquad\square$

Example 2.16 Let $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$ and a, b incomparable. The operations $\otimes$ and $\rightarrow$ are defined as follows:

⊗	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

→	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then L becomes a residuated lattice (see [1]). The filters of L are $\{1\}, \{c, 1\}, \{a, c, 1\}, \{b, c, 1\}$ and L. It is easy to check that the prime filters of L are $\{a, c, 1\}, \{b, c, 1\}$, and $O(\{a, c, 1\}) = \{1\}$, $O(\{b, c, 1\}) = \{1\}$.

Example 2.17 Let $L = \{0, a, b, 1\}$ with $0 < a, b < 1$ and a, b incomparable. The operations $\otimes$ and $\rightarrow$ are defined as follows:

⊗	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

→	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

It is routine to verify that with the above operations, L is a residuated lattice and the filters of L are $\{1\}, \{a, 1\}, \{b, 1\}$ and L. In addition, the prime filters of L are $\{a, 1\}, \{b, 1\}$ and $O(\{a, 1\}) = \{a, 1\}$, $O(\{b, 1\}) = \{b, 1\}$.

3 The sheaf representations of residuated lattices

In this section, we introduce the notion of sheaf space of residuated lattices and construct the sheaf representations of residuated lattices.

Definition 3.1 A sheaf space of residuated lattices is a triple (E, p, X) satisfying the following conditions:

(1) Both E and X are topological spaces.

(2) $p : E \longrightarrow X$ is a local homeomorphism from E onto X, i.e. for any $e \in E$, there are open neighbourhoods U and U' of e and $p(e)$ such that p maps U homeomorphically onto U'.

(3) For any $x \in X, p^{-1}(\{x\}) = E_x$ is a residuated lattice.

(4) The functions defined by $(a, b) \longmapsto a \wedge_x b, (a, b) \longmapsto a \vee_x b, (a, b) \longmapsto a \otimes_x b, (a, b) \longmapsto a \rightarrow_x b$ from the set $\{(a, b) \in E \times E | p(a) = p(b)\}$ into E are continuous, where $x = p(a) = p(b)$.

(5) The functions $\underline{0}, \underline{1} : X \longrightarrow E$ assigning to every x in X the 0_x and 1_x of E_x respectively, are continuous.

Remark 3.2 In the Definition 3.1, E is usually called the total space, X as the base space and E_x is called the stalk of E at $x \in X$.

Definition 3.3 Let (E, p, X) be a sheaf space of residuated lattices. For any $Y \subseteq X$, a function $\sigma : Y \longrightarrow E$ is called a section over Y if it is continuous such that for any $y \in Y, p(\sigma(y)) = y$.

Remark 3.4 If we define the operations pointwisely on the set of all sections over Y, it constitutes a residuated lattice. We denote it by $\Gamma(Y, E)$. The elements of $\Gamma(X, E)$ are called *global sections*.

Definition 3.5 ([19]) Suppose that L and L' are residuated lattices. A residuated lattice morphism is

a function $h : L \longrightarrow L'$ such that $h(a \wedge_L b) = h(a) \wedge_{L'} h(b), h(a \vee_L b) = h(a) \vee_{L'} h(b), h(a \otimes_L b) = h(a) \otimes_{L'} h(b), h(a \rightarrow_L b) = h(a) \rightarrow_{L'} h(b)$ and $h(0) = 0', h(1) = 1'$.

Definition 3.6 A sheaf representation of a residuated lattice L will mean an injective residuated lattice morphism $\phi : L \longrightarrow \Gamma(X, E)$ from L to the residuated lattice $\Gamma(X, E)$ of global sections of a sheaf space of residuated lattices (E, p, X).

Lemma 3.7 *Let (E, p, X) be a sheaf space of residuated lattices. If we define $p_x : \Gamma(X, E) \longrightarrow E_x$ by $p_x(\sigma) = \sigma(x)$, then for any $x \in X$, p_x is a residuated lattice morphism.*

Proof. Here we only prove that $\otimes$ is a morphism, the proofs for other operations are similar. For any $\sigma, \mu \in \Gamma(X, E)$, $p_x(\mu \otimes \sigma) = (\mu \otimes \sigma)(x) = \mu(x) \otimes_x \sigma(x) = p_x(\mu) \otimes_x p_x(\sigma)$, and $p_x(\underline{0}) = \underline{0}(x) = 0_x, p_x(\underline{1}) = \underline{1}(x) = 1_x$. □

Lemma 3.8 *If L is a residuated lattice, then for any $a \in L, V(a) = \{P \in \mathrm{Spec}(L) | a \in O(P)\}$ is open in $\mathrm{Spec}(L)$.*

Proof. Assume that $P \in V(a)$, then $a \in O(P)$. Thus there exists $b \in L - P$ such that $a \vee b = 1$. If $Q \in D(b)$, then $b \notin Q$. Hence $a \in O(Q)$, i.e. $Q \in V(a)$. Therefore $P \in D(b) \subseteq V(a)$. This shows that $V(a)$ is open. □

In the sequel, we will construct a sheaf space for each residuated lattice using the residuated lattice L and the topological space $\mathrm{Spec}(L)$. Let E_L be the disjoint union of the set $\{L/O(P)\}_{P \in \mathrm{Spec}(L)}$ and $\pi : E_L \longrightarrow \mathrm{Spec}(L)$ the canonical projection.

Theorem 3.9 *Let L be a residuated lattice. Then the family $\mathcal{B} = \{D(F, a) : F \in \mathcal{F}(L)$ and $a \in L\}$ is a base for a topology on E_L, where $D(F, a) = \{a_P : P \in D(F)\}$ and $a_P = a/O(P)$.*

Proof. We complete the proof in two steps.

(i) For every $D_1, D_2 \in \mathcal{B}$ and $x \in D_1 \cap D_2$, there exists a $D \in \mathcal{B}$ such that $x \in D \subseteq D_1 \cap D_2$.

Take $D_1 = D(F_1, a), D_2 = D(F_2, b)$ with $F_1, F_2 \in \mathcal{F}(L)$ and $a, b \in L$. Suppose that $x \in D(F_1, a)$, $x \in D(F_2, b)$, then there exists $P \in D(F_1)$ and $Q \in D(F_2)$ such that $x = a_P = b_Q$. Thus $P = Q$ and $(a \rightarrow b) \otimes (b \rightarrow a) \in O(P)$. Hence $P \in D(F_1) \cap D(F_2) \cap V((a \rightarrow b) \otimes (b \rightarrow a)) := W$. By Remark 2.13 and Lemma 3.8, we have that W is open in $\mathrm{Spec}(L)$. Hence there exists a filter F such that $P \in D(F) \subseteq W \subseteq D(F_1) \cap D(F_2)$. Therefore $D(F, a) = \{a_p : P \in D(F)\} \subseteq D(F_1, a)$ and $D(F, a) = \{a_P : P \in D(F)\} = \{b_P : P \in D(F)\} \subseteq \{b_P : P \in D(F_2)\} = D(F_2, b)$, because $(a \rightarrow b) \otimes (b \rightarrow a) \in O(P)$. Therefore $x \in D(F, a) \subseteq D(F_1, a) \cap D(F_2, b)$.

(ii) For every $x \in E_L$, there exists a $D \in \mathcal{B}$ with $x \in D$.

Suppose that $x \in E_L$, then there exists $a \in L$ and $P \in \mathrm{Spec}(L)$ such that $x = a_P$. Thus there exists $G \in \mathcal{F}(L)$ such that $P \in D(G)$. This shows that $x \in D(G, a)$. □

In the sequel, we will use $\mathcal{T}(\mathcal{B})$ to denote the topology on E_L generated by the above $\mathcal{B}$.

Theorem 3.10 *The assignment $\pi : E_L \longrightarrow \mathrm{Spec}(L)$ defined by $a_P \longmapsto P$ is a local homeomorphism of $(E_L, \mathcal{T}(\mathcal{B}))$ onto $\mathrm{Spec}(L)$.*

Proof. The mapping π is well defined and it is clear that π is surjective. Suppose that $a_P \in E_L$ and $U = D(F, a)$ is an open neighbourhood of a_P from $\mathcal{B}$. Obviously, $\pi(D(F, a)) = D(F)$. The restriction π_U of π to U is injective from U into $D(F)$.

(i) π_U is continuous: In fact, suppose that $D(G)$ is an open set of $\mathrm{Spec}(L)$, then $D(F) \cap D(G) = D(F \cap G)$ is a base open set in $D(F)$. Also $\pi_U^{-1}(D(F \cap G)) = \{a_P : P \in D(F \cap G)\} = D(F \cap G, a)$ and it is an open subset of $D(F, a)$.

(ii) π_U is open: To see this, assume that $D(H, b)$ is a base open set of E_L. Then $D(H, b) \cap U$ is a base open subset of U. Also $\pi_U(U \cap D(H, b)) = D(F) \cap D(H)$, which is open in $D(F)$. □

Proposition 3.11 *For any* $a \in L$, *the function* $\hat{a} : \mathrm{Spec}(L) \longrightarrow E_L$ *defined by* $\hat{a}(P) = a_P$ *is a global section of* $(E_L, \pi, \mathrm{Spec}(L))$.

Proof. First, $\pi(\hat{a}(P)) = \pi(a_P) = P$. Next we prove that $\hat{a}$ is continuous. Actually, for any $D(F, a) \in \mathcal{B}$, $\hat{a}^{-1}(D(F, a)) = D(F)$, which is open in $\mathrm{Spec}(L)$. And for any $b \in L, b \neq a, D(F, b) \in \mathcal{B}$,

$$
\begin{aligned}
\hat{a}^{-1}(D(F, b)) &= D(F) \bigcap \{P | a_P = b_P\} \\
&= D(F) \bigcap \{P \in \mathrm{Spec}(L) | (a \to b) \otimes (b \to a) \in O(P)\} \\
&= D(F) \bigcap \{P \in \mathrm{Spec}(L) | a \to b \in O(P)\} \bigcap \{P \in \mathrm{Spec}(L) | b \to a \in O(P)\} \\
&= D(F) \bigcap V(a \to b) \bigcap V(b \to a)
\end{aligned}
\tag{1}
$$

By Remark 2.13 and Lemma 3.8, we know that $\hat{a}^{-1}(D(F, b))$ is open in $\mathrm{Spec}(L)$. $\qquad\square$

Corollary 3.12 *The functions* $\hat{0} : \mathrm{Spec}(L) \longrightarrow E_L$ *and* $\hat{1} : \mathrm{Spec}(L) \longrightarrow E_L$ *are global sections of* $(E_L, \pi, \mathrm{Spec}(L))$.

Let $E_L \vartriangle E_L = \bigcup \{E_P \times E_P : P \in \mathrm{Spec}(L)\}$ and equip $E_L \vartriangle E_L$ with the subspace topology of the product space $E_L \times E_L$. It is well known that a base for the topology on $E_L \times E_L$ is $\mathcal{B}' = \{D(F, a) \times D(G, b) : F, G \in \mathcal{F}(L) \text{ and } a, b \in L\}$. Thus a base for the induced topology on $E_L \vartriangle E_L$ is given by $\mathcal{B}'' = \{(B(a, b), F) : F \in \mathcal{F}(L) \text{ and } a, b \in L\}$, where $(B(a, b), F)$ is the set $\{(a_P, b_P) : P \in D(F)\}$.

Proposition 3.13 *For any* $P \in \mathrm{Spec}(L)$, *the functions* $(a_P, b_P) \longmapsto a_P \wedge_P b_P, (a_P, b_P) \longmapsto a_P \vee_P b_P$, $(a_P, b_P) \longmapsto a_P \otimes_P b_P, (a_P, b_P) \longmapsto a_P \to_P b_P$ *from the set* $\{(a_P, b_P) \in E_L \times E_L | \pi(a) = \pi(b)\}$ *into* E_L *are continuous, where* $P = \pi(a) = \pi(b)$.

Proof. We only prove the continuity of the operation $\otimes_P$. The proofs for the rest of the operations are similar. Let $(a_P, b_P) \in E_L \vartriangle E_L$ and $D(F, a \otimes b)$ a neighbourhood of $(a \otimes_P b)_P$. Then $(B(a, b), F)$ is a neighbourhood of (a_P, b_P), whose image by $\otimes_P$ is contained in $D(F, a \otimes b)$. $\qquad\square$

Theorem 3.14 *For any residuated lattice* L, $(E_L, \pi, \mathrm{Spec}(L))$ *is a sheaf space of* L.

Proof. For any $P \in \mathrm{Spec}(L)$, $\pi^{-1}(\{P\}) = L/O(P)$. And for any $P \in \mathrm{Spec}(L)$, $O(P)$ is a proper filter of L, thus $L/O(P)$ is a residuated lattice. By Theorem 3.10, Proposition 3.11, Corollary 3.12 and Proposition 3.13, we deduce that $(E_L, \pi, \mathrm{Spec}(L))$ is a sheaf space of L. $\qquad\square$

Lemma 3.15 *([1]) If* F *is a filter of a residuated lattices* L *and* $a \in L - F$, *then there exists a prime filter* P *of* L *such that* $F \subseteq P$ *and* $a \notin P$.

Proposition 3.16 $\bigcap \{P | P \in \mathrm{Spec}(L)\} = \{1\}$.

Proof. Clearly $\{1\} \subseteq \bigcap \{P | P \in \mathrm{Spec}(L)\}$. Conversely assume that $a \neq 1$, then by Lemma 3.15, there is a $P \in \mathrm{Spec}(L)$ such that $a \notin P$. Thus $a \notin \bigcap \{P | P \in \mathrm{Spec}(L)\}$. Therefore $\bigcap \{P | P \in \mathrm{Spec}(L)\} \subseteq \{1\}$. $\qquad\square$

For any $P \in \mathrm{Spec}(L)$, $O(P)$ is a subset of P and $1 \in O(P)$, thus the result below follows immediately.

Corollary 3.17 $\bigcap \{O(P) | P \in \mathrm{Spec}(L)\} = \{1\}$.

Theorem 3.18 *If* L *is a residuated lattice, then the family* $\{O(P)\}_{P \in \mathrm{Spec}(L)}$ *canonically determines a sheaf representation of* L.

Proof. Define $\varphi : L \longrightarrow \Gamma(\mathrm{Spec}(L), E_L)$ by $\varphi(a) = \hat{a}$. We only prove that for any $a, b \in L, \varphi(a \otimes b) = \varphi(a) \otimes_P \varphi(b)$. The proofs for rest of the operations are similar. For any $P \in \mathrm{Spec}(L)$, $\varphi(a \otimes b)(P) = \widehat{(a \otimes b)}(P) = a \otimes b/O(P) = a/O(P) \otimes_P b/O(P) = \hat{a}(P) \otimes_P \hat{b}(P) = \varphi(a)(P) \otimes_P \varphi(b)(P) = (\varphi(a) \otimes_P \varphi(b))(P)$.

Thus $\varphi(a \otimes b) = \varphi(a) \otimes_P \varphi(b)$. Next, we prove that the mapping φ is injective. Assume that $\varphi(a) = \varphi(b)$. Then for any $P \in \mathrm{Spec}(L)$, $a_P = b_P$. Thus $(a \to b) \otimes (b \to a) \in \bigcap\{O(P)|P \in \mathrm{Spec}(L)\} = \{1\}$, i.e. $a = b$.□

Problem 3.19 *For what L, is the mapping φ surjective?*

4   Conclusions and future work

In this paper, we investigate the properties of the family of all the prime filters of residuated lattices. Based on this, we construct the sheaf space of residuated lattices and obtain a sheaf representation of residuated lattices.

In [6], Ferraioli and Lettieri took the primary ideals as the corresponding ideals of the prime ideals and proved that every MV-algebra and the MV-algebra of all global sections of its sheaf space are isomorphic. In [8,9], the scholars proved every MV-algebra A is isomorphic to the MV-algebra of global sections of a sheaf F of MV-algebras with stalks that are linear. In our future work, we will investigate when these results hold for a residuated lattice, specifically, for what residuated lattice L, the mapping $\varphi : L \longrightarrow \Gamma(\mathrm{Spec}(L), E_L)$ is surjective. For example, is φ surjective for any Heyting algebra L?

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