Further Studies on Open Well-filtered Spaces

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Abstract

The open well-filtered spaces were introduced by Shen, Xi, Xu and Zhao to answer the problem whether every core-compact well-filtered space is sober. In the current paper we explore further properties of open well-filtered spaces. One of the main results is that if a space is open well-filtered, then so is its upper space (the set of all nonempty saturated compact subsets equipped with the upper Vietoris topology). Some other properties on open well-filtered spaces are also studied.

Keywords: well-filtered space, sober space, core-compact space, locally compact space, open well-filtered space

1 Introduction

The open well-filtered spaces were introduced in [6] and used to give an alternative and more natural proof of the conjectured conclusion that every core compact and well-filtered space is sober. In [6], the authors actually proved a stronger conclusion: every core-compact and open well-filtered space is sober, as every well-filtered space is open well-filtered and the converse conclusion is not true in general. By [8] and [10], one knows that a space is well-filtered if and only if its upper space is well-filtered. It is thus natural to wonder whether a similar result holds for open well-filteredness. In this paper we shall give a partial answer to this problem and prove that if a space is open well-filtered then so is its upper space. A number of new results on open well-filtered spaces will also be presented here.

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2   Preliminaries

We first recall some basic definitions and results that will be used in the paper.

Let P be a poset. A nonempty subset D of P is *directed* (resp., *filtered*) if every two elements of D have an upper (resp., lower) bound in D. A poset P is a *directed complete poset*, or a *dcpo* for short, if for any directed subset $D \subseteq P$, the supremum $\bigvee D$ exists.

For any subset A of a poset P,

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\}, \text{ and } \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

In particular, for each $x \in X$, we write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$.

For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P with $\bigvee D$ existing, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\uparrow x = \{y \in P : x \ll y\}$ and $\downarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\downarrow x$ is directed and $x = \bigvee \downarrow x$. A continuous dcpo is also called a *domain*.

A subset U of a poset P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P with $\bigvee D$ existing, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology on P, called the *Scott topology* on P and denote by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

For a T_0 space X, the specialization order $\leq$ on X is defined by $x \leq y$ iff $x \in \mathrm{cl}_X(\{y\})$, where cl_X is the closure operator of X. Clearly, $\mathrm{cl}_X(\{y\}) = \downarrow y$.

In what follows, if no otherwise specified, the partial order on a T_0 space will mean the specialization order.

Remark 2.1 [1,2] For each poset $(P, \leq_P)$, the specialization order on ΣP coincides with $\leq_P$.

For any T_0 space X, we shall often use $\mathcal{O}(X)$ to denote the topology of X (the collection of all open sets of X). For any subset A of X, the *saturation* of A, denoted by $Sat_X(A)$, is defined by

$$Sat_X(A) = \bigcap \{U \in \mathcal{O}(X) : A \subseteq U\},$$

or equivalently, $Sat_X(A) = \uparrow A$ with respect to the specialization order (see [2, Proposition 4.2.9]). A subset A of X is *saturated* if $A = Sat_X(A)$.

For any $U, V \in \mathcal{O}(X)$, we write $U \ll V$ for that U is way-below V in the poset $(\mathcal{O}(X), \subseteq)$. Using a similar proof to that of the Alexander's Subbase Lemma (see [2, Theorem 4.4.29]), we can obtain the following result.

Lemma 2.2 *Let X be a T_0 space, $\mathcal{S}$ be a subbase for $\mathcal{O}(X)$, and $U, V \in \mathcal{O}(X)$. Then $U \ll V$ if and only if one can extract a finite subcover of U from any cover $\{U_i : i \in I\} \subseteq \mathcal{S}$ of V.*

Definition 2.3 [6] Let X be a T_0 space.

(1) A subfamily $\mathcal{F} \subseteq \mathcal{O}(X)$ is called $\ll$-*filtered*, denoted by $\mathcal{F} \subseteq_{flt} \mathcal{O}(X)$, if for any $U_1, U_2 \in \mathcal{F}$, there exists $U_3 \in \mathcal{F}$ such that $U_3 \ll U_1$ and $U_3 \ll U_2$.

(2) X is called *open well-filtered* if for each $\ll$-filtered family $\mathcal{F} \subseteq \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies that $V \subseteq U$ for some $V \in \mathcal{F}$.

Proposition 2.4 *[6] If X is an open well-filtered space and $\{U_i : i \in I\}$ is a $\ll$-filtered family of nonempty open sets, then $\bigcap\{U_i : i \in I\}$ is a nonempty compact saturated set.*

Definition 2.5 [1,2] A T_0 space X is called *core-compact* if for each $x \in X$ and $U \in \mathcal{O}(X)$ such that $x \in U$, there exists $V \in \mathcal{O}(X)$ such that $x \in V \ll U$.

Remark 2.6 [1,2] A T_0 space X is core-compact if and only if the poset $(\mathcal{O}(X), \subseteq)$ is continuous.

Theorem 2.7 *[6] Every core-compact open well-filtered space is sober.*

Definition 2.8 [1,2] A T_0 space X is called *well-filtered* if for any filtered family $\{Q_i : i \in I\}$ of compact saturated subsets of X and any open set $U \subseteq X$, $\bigcap_{i \in I} Q_i \subseteq U$ implies $Q_{i_0} \subseteq U$ for some $i_0 \in I$.

Definition 2.9 [1,2] A nonempty subset A of a topological space X is called *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called *sober*, if for any irreducible closed set F of X, there is a (unique) point $x \in X$ such that $F = \mathrm{cl}_X(\{x\})$.

The following result on irreducible sets in product spaces will be used in the sequel.

Proposition 2.10 *[2, Proposition 8.4.7] Let $\{X_i : i \in I\}$ be a family of topological spaces. The irreducible closed sets in $\prod_{i \in I} X_i$ are exactly the sets of the form $\prod_{i \in I} C_i$, where each C_i is an irreducible closed set in X_i $(i \in I)$.*

The relations among well-filtered spaces, open well-filtered spaces and sober spaces are shown below:
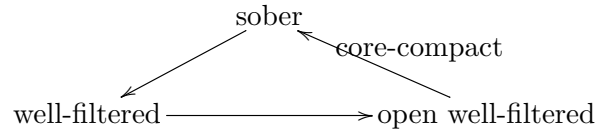
The following lemma will be used in the sequel.

Lemma 2.11 *[6, Lemma 2.4] Let X be a T_0 space, $\{U_i : i \in I\}$ be a $\ll$-filtered family of open sets of X, and F be a closed set of X. If $F \cap U_i \neq \emptyset$ for all $i \in I$, then there is a minimal closed set $F_0 \subseteq F$ such that $F_0 \cap U_i \neq \emptyset$ for all $i \in I$. In addition, this F_0 is irreducible.*

3 A new characterization for open well-filtered spaces

In [7,9], a characterization for well-filtered spaces by means of KF-sets is obtained. We now prove a similar characterization for open well-filtered spaces.

Definition 3.1 A nonempty subset A of a T_0 space X is called an *open well-filtered set*, or *OWF-set* for short, if there exists a $\ll$-filtered family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ such that $\mathrm{cl}(A)$ is a minimal closed set that intersects all U_i, $i \in I$.

Denote by $\mathsf{OWF}(X)$ the set of all closed OWF-subsets of X.

Remark 3.2 (1) A subset of a topological space is an OWF-set if and only if its closure is an OWF-set.

(2) By Lemma 2.11, every OWF-set is irreducible.

Theorem 3.3 *Let X be a T_0 space. Then the following statements are equivalent:*

(1) *X is open well-filtered;*

(2) *$\forall A \in \mathsf{OWF}(X)$, there exists a unique $x \in X$ such that $A = \mathrm{cl}(\{x\})$.*

Proof. $(1) \Rightarrow (2)$. Let $A \in \mathsf{OWF}(X)$. Then there exists $\{U_i : i \in I\} \subseteq_{flt} \mathcal{O}(X)$ such that A is a minimal closed set that intersects all U_i, $i \in I$. Since X is open well-filtered, it follows that $\bigcap_{i \in I} U_i \cap A \neq \emptyset$. Choose one $x \in \bigcap_{i \in I} U_i \cap A$. Then $\mathrm{cl}(\{x\}) \subseteq A$, and it is a closed set that intersects all U_i, $i \in I$. By the minimality of A, we have that $A = \mathrm{cl}(\{x\})$. The uniqueness of x is determined by the T_0 separation of X.

$(2) \Rightarrow (1)$. Let $\{U_i : i \in I\} \subseteq_{flt} \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$ such that $\bigcap_{i \in I} U_i \subseteq U$. We need to show that $U_i \subseteq U$ for some $i \in I$. If, on the contrary, that $U_i \not\subseteq U$ for all $i \in I$, then by Lemma 2.11, there exists a minimal (irreducible) closed set $A \subseteq X \setminus U$ that intersects all U_i, $i \in I$. Thus $A \in \mathsf{OWF}(X)$. By the assumption, there exists a unique $x \in X$ such that $A = \mathrm{cl}(\{x\})$. For each $i \in I$, since $\mathrm{cl}(\{x\}) \cap U_i = A \cap U_i \neq \emptyset$, it follows that $x \in U_i$, which implies that $x \in \bigcap_{i \in I} U_i \subseteq U$. Thus $x \in U \cap A$, which contradicts $A \subseteq X \setminus U$. This contradiction completes the proof. $\square$

Example 3.4 Let $\mathbb{N}^+$ be the set of all positive integers, $\mathbb{N}^+_{\mathrm{cof}}$ be the space of $\mathbb{N}^+$ with the co-finite topology (the open sets are $\emptyset$ and all the complements of finite subsets of $\mathbb{N}^+$), and let $\mathbb{N}^+_\alpha$ be the Alexandoff space of $\mathbb{N}^+$ (the open sets are the upper subsets of $\mathbb{N}^+$ with the usual order of numbers). We have the following claims:

(c1) It is trivial to check that each subset of $\mathbb{N}^+$ is compact in both $\mathbb{N}^+_{\mathrm{cof}}$ and $\mathbb{N}^+_\alpha$. We then deduce that $U \ll V$ iff $U \subseteq V$ for any open sets U, V in $\mathbb{N}^+_{\mathrm{cof}}$ or $\mathbb{N}^+_\alpha$.

(c2) Neither $\mathbb{N}^+_{cof}$ nor $\mathbb{N}^+_\alpha$ is open well-filtered.

   For each $n \in \mathbb{N}^+$, $U_n = \mathbb{N}^+ \setminus \{1, 2, 3, \ldots, n\}$ is a nonempty open set in both $\mathbb{N}^+_{\mathrm{cof}}$ and $\mathbb{N}^+_\alpha$. From (c1), it follows that the family $\{U_n : n \in \mathbb{N}^+\}$ is $\ll$-filtered, but $\bigcap_{n \in \mathbb{N}^+} U_n = \emptyset$. By Proposition 2.4, we obtain (c2).

The following example shows that a saturated subspace of an open well-filtered space need not be open well-filtered.

Example 3.5 Let $\mathbb{J} = \mathbb{N}^+ \times (\mathbb{N}^+ \cup \{\omega\})$ be the Johnstone's dcpo [2,3], which is ordered by $(m, n) \leq (m', n')$ iff either $m = m'$ and $n \leq n'$, or $n' = \omega$ and $n \leq m'$ (refer to Figure 1).
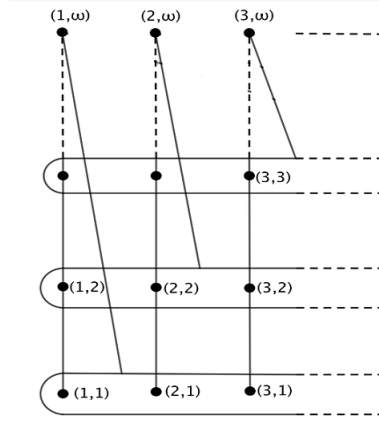

Fig. 1. The Johnstone's dcpo $\mathbb{J}$

We have the following conclusions.

(r1) $\Sigma\mathbb{J}$ is open well-filtered.

   Note that $\forall U, V \in \sigma(\mathbb{J})$, $U \ll V$ iff $U = \emptyset$ (see [2, Exercise 5.2.15]), which implies that each $\ll$-filtered family $\mathcal{F}$ of $\sigma(\mathbb{J})$ is equal to $\{\emptyset, U\}$, where U is an arbitrary Scott open set in $\mathbb{J}$. This means there exists no OWF-set in $\Sigma\mathbb{J}$. By Theorem 3.3, we deduce that $\Sigma\mathbb{J}$ is open well-filtered.

(r2) The set of maximal points $\mathbb{N}^+ \times \{\omega\}$, as a saturated subspace of $\Sigma\mathbb{J}$, is homeomorphic to $\mathbb{N}^+_{cof}$ of Example 3.4, and thus is not open well-filtered.

From (r1), we have that $\downarrow x$ (which is exactly the closure of $\{x\}$ in $\Sigma\mathbb{J}$) is not an OWF-set for each $x \in J$. We then deduce that the closure of singletons need not be OWF-sets. From (r2), it follows that the saturated subspace of an open well-filtered spaces need not be open well-filtered.

Remark 3.6 From Example 3.5, we deduce that if each $\ll$-filtered family of open sets in a space X contains the empty set, then X must be open well-filtered.

The following example shows that neither the closed subspace nor the retract of an open well-filtered space is open well-filtered in general.

Example 3.7 [6, Example 4.13] Let $P = \mathbb{J} \cup \mathbb{N}^+$, where $\mathbb{J}$ is the Johnstone's dcpo. For any $x, y \in P$, define $x \leq y$ if one of the following conditions holds (refer to Figure 2):
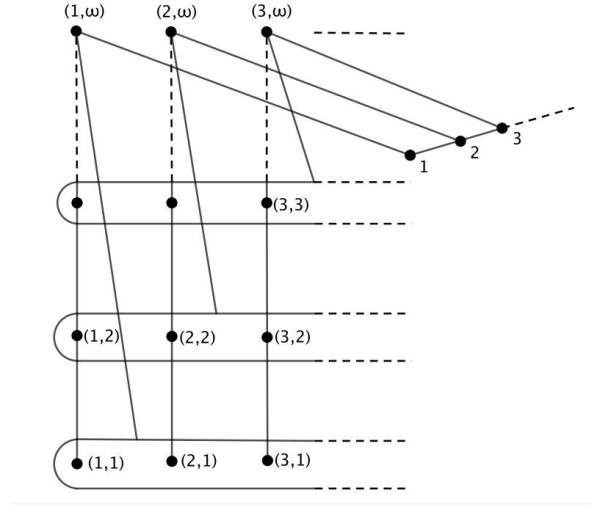
Fig. 2. The poset P of Example 3.7

(i) $x, y \in \mathbb{N}^+$ and $x \leq y$ in $\mathbb{N}^+$ with the usual ordering;

(ii) $x, y \in \mathbb{J}$ and $x \leq y$ in $\mathbb{J}$;

(iii) $x \in \mathbb{N}^+$ and $y = (x, \omega)$.

We have the following conclusions.

(1) ΣP is open well-filtered (see [6, Example 4.13] for details).

(2) $\mathbb{N}^+$ is a Scott closed subset of P, and as a subspace of P, is homeomorphic to $\mathbb{N}_\alpha^+$, hence is not open well-filtered (by Example 3.4 (2)).

(3) $\mathbb{N}^+$, as a subspace of P, is a retract of P.
   Let $e : \mathbb{N}^+ \longrightarrow P$ be the identity embedding, and define $r : P \longrightarrow \mathbb{N}^+$ as follows: $\forall x \in P, \forall n \in \mathbb{N}^+$, $r(x) = n$ iff

$$\begin{cases} x \in \downarrow(1, \omega), & \text{when } n = 1; \\ x \in \downarrow(n, \omega) \setminus \downarrow(n - 1, \omega), & \text{when } n \geq 2. \end{cases}$$

Since $\mathbb{N}^+$ is a closed set in ΣP, we have that e is continuous, and note that $r^{-1}(\downarrow n) = \downarrow(n, \omega)$ is Scott closed for each $n \in \mathbb{N}^+$, which implies that r is a continuous mapping. It is clear that the composition $r \circ e$ is the identity mapping on $\mathbb{N}^+$. Thus (3) holds.

From above (1)–(3), we deduce that the closed subspace or the retract of an open well-filtered space need not be open well-filtered.

Proposition 3.8 *Let X be a core-compact space. Then every irreducible set in X is an OWF-set.*

Proof. Suppose that A is an irreducible subset of X. Let

$$\mathcal{F} = \{U \in \mathcal{O}(X) : A \cap U \neq \emptyset\}.$$

Claim 1: $\mathcal{F} \subseteq_{flt} \mathcal{O}(X)$.
   Let $U_1, U_2 \in \mathcal{F}$. Then $A \cap U_1 \neq \emptyset$ and $A \cap U_2 \neq \emptyset$, implying that $A \cap U_1 \cap U_2 \neq \emptyset$. Take $x \in A \cap U_1 \cap U_2$. Since X is core-compact, there exists $U_3 \in \mathcal{O}(X)$ such that $x \in U_3 \ll U_1 \cap U_2$. Note that $x \in U_3 \cap A \neq \emptyset$, implying that $U_3 \in \mathcal{F}$. Thus $\mathcal{F}$ is $\ll$-filtered.

Claim 2: $\mathrm{cl}(A)$ is a minimal closed set that intersects all members of $\mathcal{F}$.
   Suppose B is a closed set such that $B \subseteq \mathrm{cl}(A)$ and $B \cap U \neq \emptyset$ for all $U \in \mathcal{F}$. We need to prove

$\mathrm{cl}(A) \subseteq B$. Otherwise, $\mathrm{cl}(A) \nsubseteq B$, which implies $A \cap (X \setminus B) \neq \emptyset$. Thus $X \setminus B \in \mathcal{F}$, which contradicts that B intersects all members of $\mathcal{F}$. Therefore, $\mathrm{cl}(A) = B$.

All this shows that $\mathrm{cl}(A) \in \mathsf{OWF}(X)$.   □

The following result is immediate by using Proposition 3.8.

Corollary 3.9 *[4,9] Every core-compact open well-filtered space is sober.*

The following example shows that the continuous image of an OWF-set need not be an OWF-set.

Example 3.10 Let X be the Alexandroff space of the Johnstone's dcpo $\mathbb{J}$ (whose open sets are the upper sets), and let $f : X \longrightarrow \Sigma\mathbb{J}$ be the identity mapping. Clearly, f is a continuous mapping. Note that for any $x \in \mathbb{J}$, since X is locally compact (hence core-compact), by Proposition 3.8, $\downarrow x$ is an OWF-set in X, but it is not an OWF-subset of $\Sigma\mathbb{J}$ by Example 3.5 (r1).

For a T_0 space X and $Y \in \mathcal{O}(X)$, if $U, V \in \mathcal{O}(Y)$, then we have that $U, V \in \mathcal{O}(X)$, and that $U \ll V$ in $(\mathcal{O}(Y), \subseteq)$ if and only if $U \ll V$ in $(\mathcal{O}(X), \subseteq)$. Using this fact, one can prove the following result easily.

Proposition 3.11 *Every open subspace of an open well-filtered space is also open well-filtered.*

Definition 3.12 *[2] A T_0 space X is called core-coherent if for any $U, V, W \in \mathcal{O}(X)$, $U \ll V$ implies that $U \cap W \ll V \cap W$.*

Example 3.5 shows that singletons need not be OWF-sets. As a remedy, we have the following result.

Theorem 3.13 *Let X be a core-coherent space. Then the following statements are equivalent:*

(1) *X is core-compact;*

(2) *all irreducible sets in X are OWF-sets;*

(3) *the singletons are OWF-sets.*

Proof. By Proposition 3.8, that (1) $\Rightarrow$ (2) is clear, and since singletons are irreducible, we obtain that (2) $\Rightarrow$ (3). Now we prove (3) $\Rightarrow$ (1).

Let $x \in X$ and U be an open neighborhood of x. Since $\{x\}$ is an OWF-set, there exists a $\ll$-filtered family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ such that $\mathrm{cl}(\{x\})$ is a minimal closed set that intersects all U_i, $i \in I$. Fix an $i_0 \in I$. It follows that $x \in U_{i_0}$. Then there exists $i_1 \in I$ such that $x \in U_{i_1} \ll U_{i_0}$. Since X is core-coherent, it holds that $x \in U \cap U_{i_1} \ll U \cap U_{i_0} \subseteq U$. Therefore, X is core-compact.   □

It is easy to verify the following lemma.

Lemma 3.14 *Let $f : X \longrightarrow Y$ be a continuous open mapping between T_0 spaces, and $U, V \in \mathcal{O}(X)$. If $U \ll V$, then $f(U) \ll f(V)$.*

Regarding the product spaces, we are still not able to prove that the product of two open well-filtered spaces is a open well-filtered space. Here is a result for some special spaces.

Theorem 3.15 *For each T_0 space X, the product $X \times \Sigma\mathbb{J}$ is open well-filtered.*

Proof. Let $U, V \in \mathcal{O}(X \times \Sigma\mathbb{J})$ such that $U \ll V$. We prove that $U = \emptyset$. Note that the projection p_2 is a continuous open mapping, so $p_2(U), p_2(V) \in \sigma(\mathbb{J})$ and by Lemma 3.14, $p_2(U) \ll p_2(V)$. Thus $p_2(U) = \emptyset$, which implies that $U = \emptyset$. By Remark 3.6, $X \times \Sigma\mathbb{J}$ is an open well-filtered space.   □

The above theorem indicates that the open well-filteredness of the product of spaces does not imply the open well-filteredness of the factor spaces. In the following, we will show that the open well-filteredness of the product of finite T_0 spaces implies that one of the factor spaces is open well-filtered.

It also is trivial to verify the following lemma.

Lemma 3.16 *Let $\{X_k : 1 \leq k \leq n\}$ be a finite family of T_0 spaces, and $U_k, V_k \in \mathcal{O}(X)$ such that $U_k \ll V_k$ for $1 \leq k \leq n$. Then $\prod_{1 \leq k \leq n} U_k \ll \prod_{1 \leq k \leq n} V_k$.*

Theorem 3.17 *Let $\{X_k : 1 \leq k \leq n\}$ be a finite family of T_0 spaces, and X be their product space. If X is an open well-filtered space, then there is k_0 ($1 \leq k_0 \leq n$) such that X_{k_0} is open well-filtered.*

Proof. Suppose on the contrary that every X_k is not open well-filtered, $1 \leq k \leq n$. Then there exist a $\ll$-filtered family $\mathcal{F}_k \subseteq \mathcal{O}(X_k)$ and an open set O_k in X_k such that $\bigcap \mathcal{F}_k \subseteq O_k$, but $U \nsubseteq O_k$ for any $U \in \mathcal{F}_k$. Define

$$\mathcal{F} = \left\{ \prod_{1 \leq k \leq n} U_k : \forall k, U_k \in \mathcal{F}_k \right\}.$$

By Lemma 3.16, one can deduce that $\mathcal{F}$ is a $\ll$-filtered family of $\mathcal{O}(X)$ such that $\bigcap \mathcal{F} \subseteq \prod_{1 \leq k \leq n} O_k \in \mathcal{O}(X)$. Since X is open well-filtered, there exists $\prod_{1 \leq k \leq n} U_k \in \mathcal{F}$ (i.e., $U_k \in \mathcal{F}_k$ for $1 \leq k \leq n$) such that $\prod_{1 \leq k \leq n} U_k \subseteq \prod_{1 \leq k \leq n} O_k$. Note that each U_k ($1 \leq k \leq n$) is not empty, which follows that $U_k \subseteq O_k$ for $1 \leq k \leq n$, a contradiction. □

4 Upper spaces and open well-filteredness

In this section, we prove that if a space X is open well-filtered then its upper space (or the Smyth power space) is also open well-filtered. The proof here makes use of a technique employed in [5].

For any topological space X, we use $\mathcal{D}(X)$ to denote the set of all nonempty compact saturated subsets of X. The *upper Vietoris topology* on $\mathcal{D}(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in \mathcal{D}(X) : K \subseteq U\}$. The set $\mathcal{D}(X)$ equipped with the upper Vietoris topology is called the *upper space* or *Smyth power space* of X. Note that $\{\Diamond F : X \setminus F \in \mathcal{O}(X)\}$ is a base of the co-upper Vietoris topology, where $\Diamond F = \{K \in \mathcal{D}(X) : K \cap F \neq \emptyset\}$.

Remark 4.1 Let X be a T_0 space, and $U, U_1, U_2 \in \mathcal{O}(X)$.

(1) $\Box U_1 \subseteq \Box U_2$ if and only if $U_1 \subseteq U_2$.

(2) For any $x \notin U$, $\Diamond \mathrm{cl}(\{x\}) \cap \Box U = \emptyset$.

(3) If $\Box U \subseteq \Box U_1 \cup \Box U_2$, then $\Box U \subseteq \Box U_1$ or $\Box U \subseteq \Box U_2$ [5].

We now state and prove the main result in this section.

Theorem 4.2 *For any open well-filtered space X, the upper space $\mathcal{D}(X)$ is open well-filtered.*

Proof. Assume that X is an open well-filtered space.

Let $\{\mathcal{U}_i : i \in I\}$ be a $\ll$-filtered family of open sets $\mathcal{U}_i$ in $\mathcal{D}(X)$ and

$$\bigcap \{\mathcal{U}_i : i \in I\} \subseteq \mathcal{U}$$

for an open set $\mathcal{U}$ in $\mathcal{D}(X)$.

For each $i \in I$, let $\mathcal{U}_i = \bigcup \{\Box U_{i,t} : t \in T_i\}$, where $U_{i,t} \in \mathcal{O}(X)$.

Assume that for each $i \in I$, $\mathcal{U}_i \nsubseteq \mathcal{U}$. Let $\mathcal{C} = \mathcal{D}(X) \setminus \mathcal{U}$. Then $\mathcal{C}$ is a closed set in $\mathcal{D}(X)$ such that $\mathcal{C} \cap \mathcal{U}_i \neq \emptyset$ for all $i \in I$. By Lemma 2.11, there is a minimal closed set $\mathcal{C}_0 \subseteq \mathcal{C}$ which also has a nonempty intersection with every $\mathcal{U}_i$.

For each $i \in I$, let

$$\hat{\mathcal{U}}_i = \bigcup \{\Box U_{i,t} : \Box U_{i,t} \cap \mathcal{C}_0 \neq \emptyset, t \in T_i\}.$$

Fact 1: If $\mathcal{U}_{i_1} \ll \mathcal{U}_{i_2} \ll \mathcal{U}_{i_3}$, then $\hat{\mathcal{U}}_{i_1} \ll \hat{\mathcal{U}}_{i_3}$. Therefore, $\{\hat{\mathcal{U}}_i : i \in I\}$ is $\ll$-filtered.

We just need to verify the first statement. To see this, let $\{\mathcal{V}_l : l \in L\}$ be a directed open cover of $\hat{\mathcal{U}}_{i_3}$. Then $\{\mathcal{V}_l \cup (\mathcal{D}(X) \setminus \mathcal{C}_0) : l \in L\}$ is a directed open cover of $\mathcal{U}_{i_3}$. By the assumption, there is a $l_0 \in L$ such that $\mathcal{U}_{i_2} \subseteq \mathcal{V}_{l_0} \cup (\mathcal{D}(X) \setminus \mathcal{C}_0)$.

By $\mathcal{U}_{i_1} \ll \mathcal{U}_{i_2}$ and the structure of the upper Vietoris topology, there exist

$$\Box W_1, \Box W_2, \cdots, \Box W_n \text{ contained in } \mathcal{V}_{l_0},$$

$$\Box G_1, \Box G_2, \cdots, \Box G_m \text{ contained in } \mathcal{D}(X) \setminus \mathcal{C}_0$$

such that

$$\mathcal{U}_{i_1} \subseteq \bigcup \{\Box W_k : 1 \le k \le n\} \cup \bigcup \{\Box G_h : 1 \le h \le m\}.$$

Note that $\hat{\mathcal{U}}_{i_1} \subseteq \mathcal{U}_{i_1}$. For each $t \in T_{i_1}$ such that $\Box U_{i_1,t} \cap \mathcal{C}_0 \ne \emptyset$, by Remark 4.1, $\Box U_{i_1,t} \subseteq \Box W_k$ for some $1 \le k \le n$, or $\Box U_{i_1,t} \subseteq \Box G_h$ for some $1 \le h \le m$. But $\Box G_h \cap \mathcal{C}_0 = \emptyset$, hence $\Box U_{i_1,t} \subseteq \Box W_k \subseteq \mathcal{V}_{l_0}$ for some k. It then follows that

$$\hat{\mathcal{U}}_{i_1} \subseteq \mathcal{V}_{l_0}.$$

Therefore $\hat{\mathcal{U}}_{i_1} \ll \hat{\mathcal{U}}_{i_3}$ holds.

For each $i \in I$, let

$$U_i = \bigcup \{U_{i,t} : \Box U_{i,t} \cap \mathcal{C}_0 \ne \emptyset\}.$$

Fact 2: If $\hat{\mathcal{U}}_{i_1} \ll \hat{\mathcal{U}}_{i_2}$, then $U_{i_1} \ll U_{i_2}$. Hence $\{U_i : i \in I\}$ is a $\ll$-filtered family of open sets in X.

As a matter of fact, it is easy to see that if $\{W_j : j \in J\} \subseteq \mathcal{O}(X)$ is a directed open cover of U_{i_2}, then $\{\Box W_j : j \in J\}$ is a directed open cover of $\hat{\mathcal{U}}_{i_2}$, hence there is $j_0 \in J$ such that $\hat{\mathcal{U}}_{i_1} \subseteq \Box W_{j_0}$, thus $U_{i_1} \subseteq W_{j_0}$.

Let

$$K = \bigcap \{U_i : i \in I\}.$$

By Proposition 2.4, K is a nonempty saturated compact set, that is $K \in \mathcal{D}(X)$.

Fact 3: $K \notin \mathcal{U}$.

Indeed, if $K \in \mathcal{U}$, then there is an open set E of X such that $K \in \Box E \subseteq \mathcal{U}$. Then $K = \bigcap \{U_i : i \in I\} \subseteq E$, and since X is open well-filtered, there is i_0 such that $U_{i_0} \subseteq E$.

Choose one U_{i_0,t_0} such that $\Box U_{i_0,t_0} \cap \mathcal{C}_0 \ne \emptyset$. Then $U_{i_0,t_0} \subseteq U_{i_0} \subseteq E$, and it follows that

$$\emptyset \ne \Box U_{i_0,t_0} \cap \mathcal{C}_0 \subseteq \Box E \cap \mathcal{C} \subseteq \mathcal{U} \cap \mathcal{C} = \emptyset,$$

a contradiction.

Fact 4: $K \in \bigcap \{\hat{\mathcal{U}}_i : i \in I\}$.

As a matter of fact, if the statement is not true, then there is $i_0 \in I$ such that $K \notin \hat{\mathcal{U}}_{i_0}$. By the definition of $\hat{\mathcal{U}}_{i_0}$. Take any U_{i_0,t_0} with $\Box U_{i_0,t_0} \cap \mathcal{C}_0 \ne \emptyset$. Then $K \not\subseteq U_{i_0,t_0}$. Take an $e \in K \setminus U_{i_0,t_0}$ and let $\mathcal{F} = \Diamond \text{cl}(\{e\})$. Then by Remark 4.1 $\mathcal{F} \cap \Box U_{i_0,t_0} = \emptyset$.

We show that $\mathcal{C}_0 \cap \mathcal{F} \cap \mathcal{U}_i \ne \emptyset$ for all $i \in I$. For this, it is enough to show that for any $i \in I$, there is a $t_i \in T_i$ such that

$$\Box U_{i,t_i} \cap \mathcal{F} \cap \mathcal{C}_0 \ne \emptyset.$$

If not, there exists $i_1 \in I$ such that

$$\Box U_{i_1,t} \subseteq (\mathcal{D}(X) \setminus \mathcal{F}) \cup (\mathcal{D}(X) \setminus \mathcal{C}_0)$$

for all $t \in T_{i_1}$. Choose i_2 such that $\hat{\mathcal{U}}_{i_2} \ll \hat{\mathcal{U}}_{i_1}$. Then there are

$$\Box V_1^1, \Box V_2^1, \cdots, \Box V_m^1 \text{ contained in } \mathcal{D}(X) \setminus \mathcal{C}_0,$$

and

$$\Box V_1^2, \Box V_2^2, \cdots, \Box V_n^2 \text{ contained in } \mathcal{D}(X) \setminus \mathcal{F}$$

such that

$$\hat{\mathcal{U}}_{i_2} \subseteq \bigcup \{\Box V_k^1 : 1 \le k \le m\} \cup \bigcup \{\Box V_l^2 : 1 \le l \le n\}.$$

By the definition of K, we have that $e \in K \subseteq U_{i_2}$, and then there is $t' \in T_{i_2}$ such that $e \in U_{i_2,t'}$ where $\Box U_{i_2,t'} \cap \mathcal{C}_0 \ne \emptyset$. Thus $Sat_X(\{e\}) \in \mathcal{F} \cap \Box U_{i_2,t'} \ne \emptyset$.

Now $\Box U_{i_2,t'} \subseteq \hat{\mathcal{U}}_{i_2} \subseteq \bigcup\{\Box V_k^1 : 1 \leq k \leq m\} \cup \bigcup\{\Box V_l^2 : 1 \leq l \leq n\}$. By Remark 2.1, we have that

$$\Box U_{i_2,t'} \subseteq \Box V_k^1 \text{ for some } 1 \leq k \leq m, \text{ or } \Box U_{i_2,t'} \subseteq \Box V_l^2 \text{ for some } 1 \leq l \leq n.$$

But for any k, since $\Box U_{i_2,t'} \cap \mathcal{C}_0 \neq \emptyset$ and $\Box V_k^1 \cap \mathcal{C}_0 = \emptyset$, it follows that $\Box U_{i_2,t'} \not\subseteq \Box V_k^1$, and for any l, since $\Box U_{i_2,t'} \cap \mathcal{F} \neq \emptyset$ and $\Box V_l^2 \cap \mathcal{F} = \emptyset$, it follows that $\Box U_{i_2,t'} \not\subseteq \Box V_l^2$, a contradiction. This shows that $\mathcal{C}_0 \cap \mathcal{F} \cap \hat{\mathcal{U}}_i \neq \emptyset$ for all $i \in I$.

By the minimality of $\mathcal{C}_0$, we have that $\mathcal{C}_0 \subseteq \mathcal{F}$. Then $\Box U_{i_0,t_0} \cap \mathcal{C}_0 \subseteq \Box U_{i_0,t_0} \cap \mathcal{F}$. But, as we pointed out earlier, $\Box U_{i_0,t_0} \cap \mathcal{F} = \emptyset$, thus $\Box U_{i_0,t_0} \cap \mathcal{C}_0 = \emptyset$. This contradicts that $\Box U_{i_0,t_0} \cap \mathcal{C}_0 \neq \emptyset$.

All these together show that $K \in \bigcap\{\hat{\mathcal{U}}_i : i \in I\}$.

Now, Fact 3 and Fact 4 contradict the assumption

$$\bigcap\{\mathcal{U}_{i \in I} : i \in I\} \subseteq \mathcal{U}.$$

The proof is thus completed. □

5   Summary

In this paper, we mainly considered the preservation of the open well-filteredness by some standardly constructed spaces from an open well-filtered space. The table below summarizes the main results, where "sp." denotes "subspaces".

open sp.	closed sp.	saturated sp.	retract	upper space	product
✓	✗	✗	✗	✓	?

Note that in many other cases, the ground spaces usually inherit the corresponding properties of their upper spaces as they can be embedded into the upper spaces under the principle filter mappings. At the moment, the following problem is still open.

Problem 5.1 *Is it true that a space is open well-filtered if its upper space is open well-filtered?*

References

[1] Gierz, G., K. Hofmann, K. Keimel, J. Lawson, M. Mislove and D. Scott, "Continuous Lattices and Domains," Encyclopedia of Mathematics and Its Applications, volume **93**, Cambridge University Press, 2003. ISBN: 9780521803380

[2] Goubault-Larrecq, J., "Non-Hausdorff Topology and Domain Theory," Cambridge University Press, 2013. ISBN: 978-1-107-03413-6
https://doi.org/10.1017/CBO9781139524438

[3] Johnstone, P. T., *Scott is not always sober*, In: Banaschewski B., Hoffmann R. E. (eds) Continuous Lattices, Springer, Berlin, Heidelberg. Lecture Notes in Mathematics, vol. **871** (1981), 282–283
https://doi.org/10.1007/BFb0089911

[4] Lawson, J., G. Wu, and X. Xi, *Well-filtered spaces, compactness, and the lower topology*, Houston J. Math. **46** (1) (2020), 283–294,
https://www.math.uh.edu/~hjm/restricted/pdf46(1)/12lawson_wu_xi.pdf

[5] Lyu, Z., Y. Chen, and X. Jia, *Core-compactness, consonance and the Smyth powerspaces*, Topo. Appl. **312** (1) (2022), 108066
https://doi.org/10.1016/j.topol.2022.108066

[6] Shen, C., X. Xi, X. Xu, and D. Zhao, *On open well-filtered spaces*, Log. Methods Comput. Sci. **16** (4) (2020), 18:1–18:10
https://doi.org/10.23638/LMCS-16(4:18)2020

[7] Shen, C., X. Xi, X. Xu, and D. Zhao, *On well-filtered reflections of T_0 spaces*, Topol. Appl. **267** (2019), 106869
    https://doi.org/10.1016/j.topol.2019.106869

[8] Xi, X., and D. Zhao, *Well-filtered spaces and their dcpo models* Math. Structures Comput. Sci. **27** (4) (2017), 507–515
    https://doi.org/10.1017/S0960129515000171

[9] Xu, X., C. Shen, X. Xi, and D. Zhao, *On T_0 spaces determined by well-filtered spaces*, Topo. Appl., **282** (15) (2020), 107323
    https://doi.org/10.1016/j.topol.2020.107323

[10] Xu, X., X. Xi, and D. Zhao, *A complete Heyting algebra whose Scott space is non-sober*, Fund. Math. **252** (3) (2021), 315–323
    https://doi.org/10.4064/fm704-4-2020