# A Hofmann-Mislove Theorem for $c$-well-filtered Spaces

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**Abstract**

The Hofmann-Mislove theorem states that in a sober space, the nonempty Scott open filters of its open set lattice correspond bijectively to its compacts saturated sets. In this paper, the concept of $c$-well-filtered spaces is introduced. We show that a retract of a $c$-well-filtered space is $c$-well-filtered and a locally Lindelöf and $c$-well-filtered $P$-space is countably sober. In particular, we obtain a Hofmann-Mislove theorem for $c$-well-filtered spaces.

*Keywords:* Hofmann-Mislove theorem, well-filtered spaces, Scott open, countably sober

## 1    Introduction

The Hofmann-Mislove Theorem plays an important role in the study of the basic topological theorems concerning sober spaces and illustrates the close relationship between domain theory and topology. It states that there exists a bijection between the nonempty Scott open filters on the open set lattice for a sober space $X$ and the compact saturated subsets ([1,5]). Moreover, it also declares that there is a bijection between the family of nonempty Scott open filters of the compact saturated sets and the open set lattice in a locally compact sober space $X$.

In recent years, some researchers have generalized the Hofmann-Mislove Theorem to some other topological spaces ([7,8,13,15,16,17]). For example, A. Jung gave an analogy result of the Hofmann-Mislove theorem for bisober spaces in [7]. In [13], M. Schröder proved that there exists a continuous retraction from the family $\mathcal{OO}_+(X)$ of all nonempty $\sigma$-Scott-open collections of open sets to the upper space $\mathcal{K}(X)$ of all countably-compact sets in a sequentially Hausdorff sequential space $X$. J.B. Yang and J.M. Shi obtained that in a countably sober space, a Scott open countable filter of open set lattice is precisely a compact saturated set in [17]. The motivation of this paper is to establish the relationship between the open set lattice and the set of all $\sigma$-Scott-open countable filters of saturated Lindelöf sets in a $c$-well-filtered space.

The remaining parts of this paper are organized as follows. Section 2 recalls some basic concepts and results used in this paper. In Section 3, we define a new notion of $c$-well-filtered spaces and investigate

its basic properties. Particularly, we prove that $c$-well-filteredness is hereditary for saturated subsets and a retract of a $c$-well-filtered space is $c$-well-filtered. In Section 4, we obtain a Hofmann-Mislove Theorem for $c$-well-filtered spaces, which states that there is a bijection between the open set lattice and the set of all $\sigma$-Scott-open countable filters of saturated Lindelöf sets.

## 2   Preliminaries

We refer to [1,4,10] for the standard definitions and notations of order theory and domain theory, and to [2,3,11] for topology.

Let $X$ be a set. We denote the family of all finite subsets (resp., countable subsets) of $X$ by $\mathrm{Fin}X$ (resp., $\mathrm{Count}X$). Let $L$ be a poset. A nonempty subset $D \subseteq L$ is *countably directed* if for every $E \in \mathrm{Count}D$, there exists $d \in D$ such that $E \subseteq \downarrow d$. *Countably filtered* is defined dually. A nonempty subset $F \subseteq L$ is called a *countable filter* if it is a countably filtered upper set. A poset $L$ is called a *countably directed complete poset* if every countably directed subset $D \subseteq L$ has a least upper bound $\sup D$ in $L$. An upper set $U$ of $L$ is $\sigma$-*Scott open* if for every countably directed subset $D \subseteq L$, $\sup D \in U$ implies $D \cap U \neq \emptyset$. All $\sigma$-Scott open subsets of $L$ form a topology, called the $\sigma$-*Scott topology* and denoted as $\sigma_c(L)$.

Let $L$ be a poset. The symbol $\sigma(L)$ denotes the *Scott topology* consisting of all Scott open subsets of $L$. The space $\Sigma L = (L, \sigma(L))$ is called the *Scott space* of $L$.

Let $X$ be a topological space and $\mathcal{O}(X)$ the open set lattice. A subset $A$ of $X$ is a *Lindelöf* set if each open cover $\mathcal{U}$ of $A$ has a countable subcover. We denote the set of all compact saturated (resp., saturated Lindelöf) subsets of $X$ by $\mathcal{Q}(X)$ (resp., $\mathcal{LQ}(X)$). A topological space $X$ is *well-filtered* iff for every filtered family $\mathcal{K}$ of $\mathcal{Q}(X)$ and for every open subset $U$ of $X$, if $\bigcap \mathcal{K} \subseteq U$, then $K \subseteq U$ for some $K \in \mathcal{K}$. A topological space $X$ is *locally Lindelöf* if for every open subset $U$ of $X$ and for every point $x \in U$, there exists $K \in \mathcal{LQ}(X)$ such that $x \in \mathrm{int}(K) \subseteq K \subseteq U$. A point $p \in X$ is called a *P-point* if its neighbourhood system is closed under countable intersection. A topological space $X$ is called a *P-space* if every point in $X$ is a $P$-point.

In what follows, we will give some results on Lindelöf sets.

**Proposition 2.1** *Let $X$ be a topological space. A subset $K$ of $X$ is a Lindelöf set if and only if for every countably directed family $\{U_i\}_{i \in I}$ of open subsets of $X$, if $K \subseteq \bigcup_{i \in I} U_i$, then $K \subseteq U_i$ for some $i \in I$.*

**Proof.** (If part) Let $\mathcal{U}$ be an open cover of $K$. Take $\mathcal{V} = \{U_\alpha : U_\alpha = \bigcup_{i \in \mathbb{N}} U_i, U_i \in \mathcal{U}, \alpha \in I\}$, where $I$ is an index set. Then $\mathcal{V}$ is a countably directed family of open subsets of $X$ and $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. Thus there exists $\alpha \in I$ such that $K \subseteq U_\alpha$, which means $\mathcal{U}$ has a countable subcover.

(Only if part) It is obvious by the countably directedness of $\{U_i\}_{i \in I}$.                    □

**Proposition 2.2** *Let $X$ be a topological space. A subset $K$ of $X$ is a Lindelöf set if and only if for every countably filtered family $\mathcal{C}$ of closed subsets of $X$, if $K \cap C \neq \emptyset$ for all $C \in \mathcal{C}$, then $K \cap \bigcap \mathcal{C} \neq \emptyset$.*

By Proposition 2.2, we get immediately the following corollary.

**Corollary 2.3** *Let $X$ be a topological space. If $K$ is a Lindelöf subset of $X$ and $C$ is a closed subset of $X$, then $K \cap C$ is a Lindelöf set.*

**Proposition 2.4** *Let $f : X \to Y$ be a continuous map. If $C$ is a Lindelöf subset of $X$, then $f(C)$ is a Lindelöf set of $Y$.*

**Proposition 2.5** *Let $X$ be a Lindelöf space. Then every closed subset $F$ of $X$ is a Lindelöf set.*

## 3   $c$-well-filtered spaces

In this section, we introduce a notion of $c$-well-filtered spaces and discuss its some basic properties.

**Definition 3.1** A topological space $X$ is *c-well-filtered* if for every countably filtered family $\{K_i\}_{i \in I}$ of saturated Lindelöf subsets of $X$ and each open subset $U$ with $\bigcap_{i \in I} K_i \subseteq U$, there is a $K_{i_0} \subseteq U$ for some $i_0 \in I$ .

**Proposition 3.2** *Suppose that $X$ is a countable set. Then every topological space $(X, \tau)$ is c-well-filtered.*

**Proof.** Obviously, each subset of $X$ is a Lindelöf set. Hence $\mathcal{LQ}(X) = \{\uparrow K : K \subseteq X\}$. Let $\mathcal{K}$ be a countably filtered family of $\mathcal{LQ}(X)$ and $U \in \tau$ with $\bigcap \mathcal{K} \subseteq U$. For every nonempty chain $\mathcal{C}$ in $\mathcal{K}$, there exists $K \in \mathcal{K}$ such that $K$ is a lower bound of $\mathcal{C}$ by the countably filteredness of $\mathcal{K}$. By the order-dual of Zorn's Lemma, $\mathcal{K}$ contains a minimal element $K_0$. Hence, $K_0 = \bigcap \mathcal{K} \subseteq U$. Therefore, $(X, \tau)$ is c-well-filtered. $\qquad\square$

The following example shows that c-well-filtered spaces are not always well-filtered spaces.

**Example 3.3** Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with the partial order defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq \omega$, where $\mathbb{J}$ is a well-known dcpo constructed by Johnstone in [6]. It is easy to see that $\mathbb{J}$ is countable. By Proposition 3.2, $\Sigma\mathbb{J}$ is a c-well-filtered space. But $\Sigma\mathbb{J}$ is not a well-filtered space (see [9, Example 3.1]).

Conversely, a well-filtered space may not be a c-well-filtered space.

**Example 3.4** Consider the real number set $\mathbb{R}$ with the co-countable topology $\tau_{coc}$, where $\tau_{coc} = \{U \subseteq \mathbb{R} : \mathbb{R} \setminus U$ is countable$\} \bigcup \{\emptyset\}$. It is known that the topological space $(\mathbb{R}, \tau_{coc})$ is a well-filtered $T_1$-space (see [12, Example 3.14]).

Next, we show that all subsets of $\mathbb{R}$ are saturated Lindelöf sets. Let $K$ be a subset of $\mathbb{R}$ and assume that $\mathcal{U}$ is an open cover of $K$. Given $U \in \mathcal{U}$, then there exists a countable subset $C$ of $\mathbb{R}$ such that $U = \mathbb{R} \setminus C$ and $K = (K \cap C) \cup (K \setminus C) \subseteq \bigcup \mathcal{U}$. It is obvious that $K \setminus C \subseteq \mathbb{R} \setminus C = U$. Since $K \cap C$ is countable and $K \cap C \subseteq \bigcup \mathcal{U}$, there exist countably many members of $\mathcal{U}$ whose union contains $K \cap C$. Hence, $K$ is a saturated Lindelöf set.

Finally, we show that $(\mathbb{R}, \tau_{coc})$ is not c-well-filtered. Let $\mathcal{K} = \{\mathbb{R} \setminus C : C$ is countable$\}$. Then $\mathcal{K} \subseteq \mathcal{LQ}(\mathbb{R})$. Obviously, $\mathcal{K}$ is countably filtered and $\bigcap \mathcal{K} = \emptyset$. However, there is no $K \in \mathcal{K}$ such that $K = \emptyset$. Therefore, $(\mathbb{R}, \tau_{coc})$ is not c-well-filtered.

Let $X$ be a $T_0$-space. The *specialization order* $\leq$ on $X$ is defined as $x \leq y$ if and only if $x \in \text{cl}(y)$. A $T_0$-space $X$ is called a *d-space* if $X$ is a directed complete poset under the specialization order and $\mathcal{O}(X) \subseteq \sigma(X)$.

We know that each well-filtered space is a d-space. However, the following example shows that a c-well-filtered space may not be a d-space.

**Example 3.5** Let $L = \mathbb{N} \cup \{\omega\}$ with the order $1 < 2 < \cdots < n < \cdots < \omega$. Then the Alexandroff space $\Gamma L = (L, a(L))$ is a c-well-filtered space by Proposition 3.2 but not a d-space (see [14, Proposition 6.5]).

Similar to well-filtered spaces, the following results hold for c-well-filtered spaces.

**Proposition 3.6** *A topological space $X$ is c-well-filtered if and only if for every closed subset $C$ of $X$ and each countably filtered family $\{K_i\}_{i \in I}$ of saturated Lindelöf subsets, if $C \cap K_i \neq \emptyset$ for all $i \in I$, then $\bigcap_{i \in I} K_i \cap C \neq \emptyset$.*

**Proposition 3.7** *Let $(X, \tau)$ be a c-well-filtered $T_0$-space. Then $\Omega(X)$ is a countably directed complete poset and $\tau \subseteq \sigma_c(\Omega(X))$, where $\Omega(X) = (X, \leq_\tau)$, $\leq_\tau$ is the specialization order of $(X, \tau)$.*

**Proof.** Let $D$ be a countably directed subset of $\Omega(X)$. Take $\mathcal{K} = \{\uparrow_\tau d : d \in D\}$, then $\mathcal{K}$ is a countably filtered family of saturated Lindelöf subsets of $X$ and $\bigcap \mathcal{K} \neq \emptyset$. Suppose that $D$ has no least upper bound. Then for any $x \in \bigcap \mathcal{K}$, there exists $y \in \bigcap \mathcal{K}$ such that $x \nleq y$. Hence, $x \in X \setminus \downarrow_\tau y$. So we have $x \notin cl_\tau D$ by the fact that $(X \setminus \downarrow_\tau y) \cap D = \emptyset$. Thus $\bigcap \mathcal{K} \cap cl_\tau D = \emptyset$, i.e. $\bigcap \mathcal{K} \subseteq X \setminus cl_\tau D$. By the c-well-filteredness of $X$, we have $\uparrow_\tau d \subseteq X \setminus cl_\tau D \subseteq X \setminus D$ for some $d \in D$. This is a contradiction. Therefore, $\Omega(X)$ is a countably directed complete poset.

Let $U \in \tau$ and $D$ be a countably directed subset of $\Omega(X)$ with $\sup D \in U$. Then $\bigcap_{d \in D} \uparrow_\tau d = \uparrow_\tau \sup D \subseteq U$. Thus we have $\uparrow_\tau d \subseteq U$ for some $d \in D$ by the c-well-filteredness of $X$. Therefore, $U \in \sigma_c(\Omega(X))$. $\qquad\square$

**Proposition 3.8** *Let $(X, \tau)$ be a c-well-filtered space and $A$ a saturated subset of $X$. Then the subspace $(A, \tau_A)$ is c-well-filtered.*

**Proof.** Let $\mathcal{K}$ be a countably filtered family of saturated Lindelöf subsets of $A$ and $V$ an open subset of $A$ with $\bigcap \mathcal{K} \subseteq V$. Then there exists an open subset $W$ of $X$ such that $V = W \cap A$. Hence, $\bigcap \mathcal{K} \subseteq W$. Now, we claim that $K$ is a saturated Lindelöf subset of $X$ for all $K \in \mathcal{K}$. It is easy to see that $K$ is a Lindelöf subset of $X$. So we need only to show that $K = \uparrow_X K$ for all $K \in \mathcal{K}$. Assume $x \in \uparrow_X K$. Then there exists $y \in K$ such that $y \leq x$. As $A = \uparrow_X A$, we obtain that $x \in A$. Thus $x \in K$, which implies $K = \uparrow_X K$. So $\mathcal{K}$ is a countably filtered family of $\mathcal{LQ}(X)$. By the $c$-well-filteredness of $X$, there exists $K \in \mathcal{K}$ such that $K \subseteq W$. Hence, $K \subseteq W \cap A = V$. This implies that $(A, \tau_A)$ is $c$-well-filtered. $\qquad\square$

A retract of a topological space $X$ is a topological space $Y$ such that there are two continuous mappings $r : X \to Y$ and $s : Y \to X$ such that $r \circ s = id_Y$.

**Proposition 3.9** *A retract of a $c$-well-filtered space is $c$-well-filtered.*

**Proof.** Let $X, Y$ be topological spaces. Assume that $Y$ is $c$-well-filtered and there are continuous maps $r : Y \to X$ and $s : X \to Y$ such that $r \circ s = id_X$. Let $\{K_i\}_{i \in I}$ be a countably filtered family of saturated Lindelöf subsets of $X$ and $U$ an open subset of $X$ with $\bigcap_{i \in I} K_i \subseteq U$. We know $s(K_i)$ is a Lindelöf subset of $Y$ for all $i \in I$ by Proposition 2.4. Thus $\{\uparrow s(K_i)\}_{i \in I}$ is a countably filtered family of saturated Lindelöf subsets of $Y$. Next, we show that $\uparrow r(\uparrow s(K_i)) \subseteq K_i$ for all $i \in I$. Take $x \in \uparrow r(\uparrow s(K_i))$, there exists $y \in K_i$ such that $r(s(y)) \leq x$, i.e. $y \leq x$. So we have $x \in K_i$. Therefore,

$$\bigcap_{i \in I} r(\uparrow s(K_i)) \subseteq \bigcap_{i \in I} \uparrow r(\uparrow s(K_i)) \subseteq \bigcap_{i \in I} K_i \subseteq U.$$

Then

$$\bigcap_{i \in I} \uparrow s(K_i) \subseteq \bigcap_{i \in I} r^{-1}(r(\uparrow s(K_i))) = r^{-1}(\bigcap_{i \in I} r(\uparrow s(K_i)) \subseteq r^{-1}(U).$$

By the $c$-well-filteredness of $Y$, we have $s(K_i) \subseteq \uparrow s(K_i) \subseteq r^{-1}(U)$ for some $i \in I$. Hence,

$$K_i \subseteq s^{-1}(s(K_i)) \subseteq s^{-1}(r^{-1}(U)) = (r \circ s)^{-1}(U) = U.$$

So $X$ is $c$-well-filtered. $\qquad\square$

A topological space $X$ is *countably sober* if and only if for every countably irreducible closed subset $A$ of $X$, there exists a unique element $x \in X$ such that $A = \downarrow x$ (see [17]).

**Proposition 3.10** *Let $X$ be a $P$-space. If $X$ is a locally Lindelöf and $c$-well-filtered space, then $X$ is countably sober.*

**Proof.** Let $A$ be a countably irreducible closed subset of $X$ and $\mathcal{U} = \{U \in \mathcal{O}(X) : A \cap U \neq \emptyset\}$. It is easy to show that $\mathcal{U}$ is a countable filter. Let $x \in A \cap U$. Then there exists $K_{x,U} \in \mathcal{LQ}(X)$ such that $x \in \text{int}(K_{x,U}) \subseteq K_{x,U} \subseteq U$ by the fact that $X$ is locally Lindelöf.

Now, to show the proposition take $\mathcal{F} = \{K_{x,U} : x \in A \cap U\}$. Next, we show that $\mathcal{F}$ is a countably filtered family of $\mathcal{LQ}(X)$. Suppose that $\{K_{x_i,U_i}\}_{i \in \mathcal{Z}_+}$ is a countable family of $\mathcal{F}$. Then for any $i \in \mathcal{Z}_+$, $A \cap \text{int}(K_{x_i,U_i}) \neq \emptyset$. Using the fact that $A$ is countably irreducible, we have $A \cap \bigcap_{i \in \mathcal{Z}_+} \text{int}(K_{x_i,U_i}) \neq \emptyset$. Let $W = \bigcap_{i \in \mathcal{Z}_+} \text{int}(K_{x_i,U_i})$. Since $X$ is a $P$-space, we have that $W$ is an open subset by [17, Proposition 4.3]. Hence, there exists $z \in A \cap W$, $K_{z,W} \in \mathcal{LQ}(X)$ such that $z \in \text{int}(K_{z,W}) \subseteq K_{z,W} \subseteq W$. So we have $K_{z,W} \in \mathcal{F}$ and $K_{z,W} \subseteq \bigcap_{i \in \mathcal{Z}_+} K_{x_i,U_i}$, which imply $\mathcal{F}$ is countably filtered.

Since $A \cap K_{x,U} \neq \emptyset$ for all $K_{x,U} \in \mathcal{F}$, we have $A \cap \bigcap K_{x,U} \neq \emptyset$ by Proposition 3.6. Hence, there exists $a \in A \cap \bigcap K_{x,U}$. We only need to show that $A = \downarrow a$. Suppose that there is a $x \in A$ such that $x \not\leq a$. Thus there exists an open subset $U$ of $X$ and $x \in U$ such that $a \notin U$. Since $X$ is a locally Lindelöf space, there exists $K_{x,U} \in \mathcal{LQ}(X)$ such that $x \in \text{int}(K_{x,U}) \subseteq K_{x,U} \subseteq U$. This implies that $K_{x,U} \in \mathcal{F}$. Thus $a \in K_{x,U}$, which is a contradiction. Therefore, $A = \downarrow a$. $\qquad\square$

## 4 A Hofmann-Mislove theorem

In this section, we give a Hofmann-Mislove Theorem for $c$-well-filtered spaces.

Let $L$ be a poset and $x, y$ two elements in $L$. We say $x$ is *countably way-below* $y$, written $x \ll_c y$, if for every countably directed subset $D$ of $L$ that has a least upper bound $\sup D$ above $y$, there is an element $d \in D$ such that $x \leq d$. Let $\downarrow_c x = \{y \in L : y \ll_c x\}$. A countably directed complete poset $L$ is said to be a *countably approximating poset* if $\downarrow_c x$ is countably directed and $x = \sup \downarrow_c x$ for all $x \in L$ (see [4]).

**Definition 4.1** Let $L$ be a poset. A $\sigma$-basis $B$ of $L$ is a subset of $L$ such that for every $x \in L$, the collection $B \cap \downarrow_c x$ of all elements of the $\sigma$-basis countably way-below $x$ is countably directed and $x = \sup(B \cap \downarrow_c x)$.

**Lemma 4.2** *A countably directed complete poset $L$ has a $\sigma$-basis if and only if it is countably approximating.*

**Proof.** If $L$ is countably approximating, then $L$ is a $\sigma$-basis. Conversely, assume that $L$ has a $\sigma$-basis $B$, then $x$ is the least upper bound of $B \cap \downarrow_c x$, which is a countably directed family of elements countably way-below $x$. So, it is clear that $x = \sup \downarrow_c x$. Now, we claim that $\downarrow_c x$ is countably directed. Let $E \in \mathrm{Count} \downarrow_c x$, then $e \ll_c x$ for every $e \in E$. Thus there exists $b_e \in B \cap \downarrow_c x$ such that $e \leq b_e$ for every $e \in E$. Since $B \cap \downarrow_c x$ is countably directed, there is a $b \in B \cap \downarrow_c x$ such that $b_e \leq b$ for all $e \in E$. Hence, $E \subseteq \downarrow b$ and $b \in \downarrow_c x$. This implies that $\downarrow_c x$ is countably directed and therefore we can conclude that $L$ is countably approximating. $\square$

**Proposition 4.3** *Let $L$ be a countably approximating poset with a countable $\sigma$-basis $B$. Then $(L, \sigma_c(L))$ is a $c$-well-filtered space.*

**Proof.** Suppose that $\{K_i\}_{i \in I}$ is a countably filtered family of saturated Lindelöf subsets of $L$ and $U$ is a $\sigma$-Scott open set with $\bigcap_{i \in I} K_i \subseteq U$. It is easy to show that $\{K_i \cap B\}_{i \in I}$ is a countably filtered family of saturated Lindelöf subsets of $B$. Hence, $\bigcap_{i \in I}(K_i \cap B) \subseteq U \cap B$. Then there exists $i_o \in I$ such that $K_{i_0} \cap B \subseteq U \cap B$ since $B$ with the inherited topology is $c$-well-filtered. Now, we claim that $K_{i_0} \subseteq U$. Assume that $K_{i_0} \not\subseteq U$, then there is a $t \in K_{i_0}$ such that $t \in L \backslash U$. Since $t = \sup(B \cap \downarrow_c t)$ and $L \backslash U$ is a $\sigma$-Scott closed set, we have that $B \cap \downarrow_c t \subseteq L \backslash U$. Thus, we can imply that $\uparrow y \cap L \backslash U \cap B \neq \emptyset$ for every $y \in B \cap \downarrow_c t$. And $\{\uparrow y : y \in B \cap \downarrow_c t\}$ is a countably filtered family of saturated Lindelöf subsets of $B$ since $B \cap \downarrow_c t$ is countably directed. So,

$$\bigcap_{y \in B \cap \downarrow_c t} \uparrow y \cap L \backslash U \cap B = \uparrow t \cap L \backslash U \cap B \neq \emptyset,$$

which contradicts $t \notin L \backslash U \cap B$. Therefore, $K_{i_0} \subseteq U$. So, we can conclude that $(L, \sigma_c(L))$ is a $c$-well-filtered space. $\square$

Let $L$ be a complete lattice. If $L$ is continuous, then $L$ is countably approximating. And it is clear that $\sigma(L) \subseteq \sigma_c(L)$. So, we can obtain that the following corollary.

**Corollary 4.4** *Suppose that $L$ is a continuous lattice with a countable $\sigma$-basis $B$, then $(L, \sigma(L))$ is a $c$-well-filtered space.*

Let $X$ be a topological space. We discuss some properties of saturated Lindelöf subsets of $X$.

**Proposition 4.5** *Let $X$ be a $c$-well-filtered space. Then $K = \bigcap \mathcal{C}$ is a nonempty saturated Lindelöf set for each countable filter base $\mathcal{C}$ of nonempty saturated Lindelöf subsets of $X$. Hence, $(\mathcal{LQ}(X), \supseteq)$ is a countably directed complete poset.*

**Proof.** It is easy to see that $K \neq \emptyset$ and $K$ is saturated. Thus we only need to prove that $K$ is a Lindelöf set. Let $\mathcal{U}$ be an open cover of $K$, i.e. $K \subseteq \bigcup \mathcal{U}$. Since $X$ is $c$-well-filtered, there exists $C \in \mathcal{C}$ such that $C \subseteq \bigcup \mathcal{U}$. So there exists countably many members of $\mathcal{U}$ whose union contains $K$ by the fact that $K \subseteq C$ and $C$ is a Lindelöf subset. Therefore, $K$ is a nonempty saturated Lindelöf subset. $\square$

**Proposition 4.6** *Let $X$ be a P-space.*

(i) *Let $K_1, K_2 \in \mathcal{LQ}(X)$ and consider the following statements:*
  (a) *There exists $U \in \mathcal{O}(X)$ such that $K_1 \supseteq U \supseteq K_2$, i.e. $int(K_1) \supseteq K_2$;*
  (b) *$K_1 \ll_c K_2$ in $\mathcal{LQ}(X)$.*
    *If $X$ is c-well-filtered, then $(a) \Rightarrow (b)$; if $X$ is locally Lindelöf, then $(b) \Rightarrow (a)$.*

(ii) *If $X$ is a locally Lindelöf and c-well-filtered space, then $(\mathcal{LQ}(X), \supseteq)$ is a countably approximating poset.*

**Proof.** (i) $(a) \Rightarrow (b)$ Suppose that $X$ is c-well-filtered and $\mathcal{D}$ is a countably directed subset of $\mathcal{LQ}(X)$ with $K_2 \leq \sup \mathcal{D}$. Using Proposition 4.5, we have $\sup \mathcal{D} = \bigcap \mathcal{D}$. Hence, $K_2 \supseteq \bigcap \mathcal{D}$. It follows from $(a)$ that there exists an open set $U$ of $X$ such that $K_1 \supseteq U \supseteq K_2 \supseteq \bigcap \mathcal{D}$. Thus $\bigcap \mathcal{D} \subseteq int(K_1) \subseteq K_1$. So there exists $D \in \mathcal{D}$ such that $D \subseteq int(K_1) \subseteq K_1$ by the c-well-filteredness of $X$. Therefore, $K_1 \leq D$. This implies that $K_1 \ll_c K_2$.

$(b) \Rightarrow (a)$ Let $U \in \mathcal{O}(X)$ and $K_2 \subseteq U$. Since $X$ is locally Lindelöf, there exists $K_x \in \mathcal{LQ}(X)$ such that $x \in int(K_x) \subseteq K_x \subseteq U$ for each $x \in K_2 \subseteq U$. Thus

$$K_2 \subseteq \bigcup_{x \in K_2} int(K_x) \subseteq \bigcup_{x \in K_2} K_x \subseteq U.$$

Since $K_2$ is a Lindelöf subset, there exists countably many members of $\{K_x : x \in K_2\}$ such that

$$K_2 \subseteq \bigcup_{i \in \mathcal{Z}_+} int(K_{x_i}) \subseteq \bigcup_{i \in \mathcal{Z}_+} K_{x_i} \subseteq U.$$

Now, take $Q_U = \bigcup_{i \in \mathcal{Z}_+} K_{x_i}$ and $\mathcal{D} = \{Q_U : U \in \mathcal{O}(X) \text{ and } K_2 \subseteq Q_U\}$. It is obvious that $\mathcal{D} \neq \emptyset$ and $Q_U \in \mathcal{LQ}(X)$ for all $Q_U \in \mathcal{D}$.

We claim that $\mathcal{D}$ is countably directed. Suppose that $\{Q_{U_i}\}_{i \in \mathcal{Z}_+}$ is a countable subset of $\mathcal{D}$. Then

$$K_2 \subseteq \bigcap_{i \in \mathcal{Z}_+} int(Q_{U_i}) \subseteq \bigcap_{i \in \mathcal{Z}_+} Q_{U_i} \subseteq \bigcap_{i \in \mathcal{Z}_+} U_i.$$

Let $V = \bigcap_{i \in \mathcal{Z}_+} int(Q_{U_i})$. Then $V$ is an open subset of $X$ by [17, Proposition 4.3]. From $K_2 \subseteq V$ we know that there exists $Q_V \in \mathcal{LQ}(X)$ such that

$$K_2 \subseteq int(Q_V) \subseteq Q_V \subseteq V \subseteq \bigcap_{i \in \mathcal{Z}_+} Q_{U_i}.$$

This implies $Q_V \in \mathcal{D}$ and so $\mathcal{D}$ is countably directed.

Now we prove that $\sup \mathcal{D} = K_2$. It is easy to see that $K_2$ is an upper bound of $\mathcal{D}$. Let $K$ be another upper bound of $\mathcal{D}$. Suppose that $K_2 \not\leq K$, i.e. $K_2 \not\supseteq K$. Then there exists $x \in K$ such that $\downarrow x \cap K_2 = \emptyset$. Hence, $K_2 \subseteq X \backslash \downarrow x$. So there exists $Q \in \mathcal{LQ}(X)$ such that $K_2 \subseteq int(Q) \subseteq Q \subseteq X \backslash \downarrow x$, which implies $Q \in \mathcal{D}$. This contradicts $K \subseteq Q$. Therefore, $\sup \mathcal{D} = K_2$.

Since $K_1 \ll_c K_2$, there exists $Q_U \in \mathcal{D}$ such that $K_1 \leq Q_U$. Therefore, $K_1 \supseteq Q_U \supseteq int(Q_U) \supseteq K_2$.

(ii) Let $K \in \mathcal{LQ}(X)$, $\downarrow_c K = \{Q \in \mathcal{LQ}(X) : Q \ll_c K\}$. It is easy to see that $\sup \downarrow_c K = K$. Hence, it is enough to show that $\downarrow_c K$ is countably directed. Suppose $\{Q_i\}_{i \in \mathcal{Z}_+}$ is a countable subset of $\downarrow_c K$. Then for any $i \in \mathcal{Z}_+$, there exists $U_i \in \mathcal{O}(X)$ such that $Q_i \supseteq U_i \supseteq K$ by (i). Thus $K \subseteq \bigcap_{i \in \mathcal{Z}_+} int(Q_i) \subseteq \bigcap_{i \in \mathcal{Z}_+} Q_i$. Let $V = \bigcap_{i \in \mathcal{Z}_+} int(Q_i)$. Since $X$ is a P-space, we have $V$ is an open subset of $X$ and $K \subseteq V$ by [17, Proposition 4.3]. So there exists $Q_V \in \mathcal{LQ}(X)$ such that $K \subseteq int(Q_V) \subseteq Q_V \subseteq V$. This implies that $Q_V \in \downarrow_c K$. Therefore, we conclude that $(\mathcal{LQ}(X), \supseteq)$ is a countably approximating poset by Proposition 4.5. $\square$

**Lemma 4.7** *Let $X$ be a c-well-filtered space and $U \in \mathcal{O}(X)$. Then the set*

$$\phi^{'}(U) = \{K \in \mathcal{LQ}(X) : K \subseteq U\}$$

*is a $\sigma$-Scott open countable filter in $(\mathcal{LQ}(X), \supseteq)$.*

**Proof.** It is obvious that $\phi^{'}(U)$ is an upper set. Let $\{K_i\}_{i \in I}$ be a countably directed subset of $\mathcal{LQ}(X)$ and $\sup_{i \in I} K_i = \bigcap_{i \in I} K_i \in \phi^{'}(U)$. Then $\bigcap_{i \in I} K_i \subseteq U$. Hence, there exists $i_0 \in I$ such that $K_{i_0} \subseteq U$ by the fact that $X$ is a $c$-well-filtered space. So $\phi^{'}(U)$ is a $\sigma$-Scott open subset. Now, we check that $\phi^{'}(U)$ is countably filtered. Let $\{K_i\}_{i \in \mathcal{Z}_+}$ be a countable subset of $\phi^{'}(U)$. Then $\inf_{i \in \mathcal{Z}_+} K_i = \bigcup_{i \in \mathcal{Z}_+} K_i$. Since $K_i \subseteq U$ for all $i \in \mathcal{Z}_+$, we have $\bigcup_{i \in \mathcal{Z}_+} K_i \subseteq U$. This implies that $\bigcup_{i \in \mathcal{Z}_+} K_i \in \phi^{'}(U)$. Therefore, $\phi^{'}(U)$ is a $\sigma$-Scott open countable filter. $\qquad\square$

Let $X$ be a topological space. The set of all $\sigma$-Scott open countable filters of $(\mathcal{LQ}(X), \supseteq)$ is denoted by $\mathrm{OCFilt}_\sigma((\mathcal{LQ}(X), \supseteq))$. We have the following theorem.

**Theorem 4.8** *Let $X$ be a P-space. If $X$ is locally Lindelöf and c-well-filtered, then the mapping*

$$\phi^{'} : \mathcal{O}(X) \to OCFilt_\sigma((\mathcal{LQ}(X), \supseteq)), \phi^{'}(U) = \{K \in \mathcal{LQ}(X) : K \subseteq U\}$$

*is an order isomorphism.*

**Proof.** (1) $\phi^{'}$ is surjective. Let $\mathcal{F} \in OCFilt_\sigma((\mathcal{LQ}(X), \supseteq))$. Take $U = \bigcup \mathcal{F}$. Now, we check that $U$ is an open subset of $X$. It is enough to show that $U$ is a neighborhood of $x$ for all $x \in U$. Assume $x \in U$. Then there exists $K \in \mathcal{F}$ such that $\uparrow x \subseteq K$. Using the fact that $\mathcal{F}$ is an upper set, we have $\uparrow x \in \mathcal{F}$. Let $\mathcal{D} = \{Q \in \mathcal{LQ}(X) : x \in \mathrm{int}(Q) \subseteq Q\}$. Since $X$ is locally Lindelöf, we have that $\mathcal{D} \neq \emptyset$. Now, we claim that $\mathcal{D}$ is countably directed. Let $\{Q_i\}_{i \in \mathcal{Z}_+}$ be a countable subset of $\mathcal{D}$. Then $x \in \bigcap_{i \in \mathcal{Z}_+} \mathrm{int}(Q_i) \subseteq \bigcap_{i \in \mathcal{Z}_+} Q_i$. As $X$ is a $P$-space, we have $\bigcap_{i \in \mathcal{Z}_+} \mathrm{int}(Q_i)$ is an open subset of $X$ by [17, Proposition 4.3]. So there exists $Q \in \mathcal{LQ}(X)$ such that $x \in \mathrm{int}(Q) \subseteq Q \subseteq \bigcap_{i \in \mathcal{Z}_+} \mathrm{int}(Q_i)$, which implies that $Q \in \mathcal{D}$. Thus $\sup \mathcal{D} = \bigcap \mathcal{D} = \uparrow x$. Since $\mathcal{F}$ is a $\sigma$-Scott open subset, there exists $Q \in \mathcal{D}$ such that $Q \in \mathcal{F}$. Therefore, $x \in \mathrm{int}(Q) \subseteq Q \subseteq U$, which implies that $U$ is a neighborhood of $x$.

Next, we show that $\phi^{'}(U) = \mathcal{F}$. Obviously, $\mathcal{F} \subseteq \phi^{'}(U)$. We only need to show that $\phi^{'}(U) \subseteq \mathcal{F}$. Let $K \in \phi^{'}(U)$ and $x \in K \subseteq U$. Then there exists $K_x \in \mathcal{F}$ such that $x \in \mathrm{int}(K_x) \subseteq K_x$. Hence, $K \subseteq \bigcup_{x \in K} K_x$. So there exists a countable subset $\{K_{x_i} : i \in \mathcal{Z}_+\}$ of $\{K_x : x \in K\}$ such that $K \subseteq \bigcup_{i \in \mathcal{Z}_+} K_{x_i}$ by the fact that $K$ is a Lindelöf subset. Thus $\bigcup_{i \in \mathcal{Z}_+} K_{x_i} \in \mathcal{F}$ because $\mathcal{F}$ is a countable filter. Therefore, $K \in \mathcal{F}$.

(2) $\phi^{'}$ is an order embedding, i.e. $U_1 \subseteq U_2 \Leftrightarrow \phi^{'}(U_1) \subseteq \phi^{'}(U_2)$.

($\Rightarrow$) It obviously holds.

($\Leftarrow$) Suppose that $U_1 \nsubseteq U_2$. Then there exists $u \in U_1$ such that $u \notin U_2$. Hence, $\uparrow u \in \phi^{'}(U_1)$ but $\uparrow u \notin \phi^{'}(U_2)$, which is a contradiction. Thus $U_1 \subseteq U_2$ as desired. $\qquad\square$

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