ELECTRONIC NOTES IN
THEORETICAL INFORMATICS
AND COMPUTER SCIENCE

ENTICS
https://entics.episciences.org

VOLUME 2
PROCEEDINGS OF
ISDT 2022

A Note on the Category of c-spaces

Zhenchao Lyu[a,1,2] Xiaolin Xie[a,1,3] Hui Kou[a,1,4]

[a] *Department of Mathematics*
Sichuan University
Chengdu, 610064, China

Abstract

We prove that the category of c-spaces with continuous maps is not cartesian closed. As a corollary, it follows that the category of locally finitary compact spaces with continuous maps also is not cartesian closed.

Keywords: directed space, c-space, cartesian closed, locally finitary compact

1 Introduction

Many people have been trying to extend domain theory to general topological spaces, see [6,4,7,9,2]. Directed spaces are introduced by Kou and Yu independently [12] in 2014 for generalizing the concept of Scott spaces, which is equivalent to that of T_0 monotone determined spaces introduced by Erné [5]. In the same paper Kou and Yu proved that the category of directed spaces with continuous maps (**DTop** for short) is cartesian closed. There are many important directed spaces in domain theory, for instance locally finitary compact spaces are directed spaces; in particular c-spaces and Alexandroff spaces are directed spaces. Since the category of continuous domains is not cartesian closed, and since the position of the category of c-spaces in the category of directed spaces is similar to that of continuous domains in dcpos [3], a natural question arises: Is the category of c-spaces cartesian closed? In this short note, we answer this question in the negative.

2 Preliminaries

We refer to [8,1,9] for the standard definitions and notations in order theory, topology and domain theory. A partially ordered set D is called a dcpo if every directed subset of D has a supremum in D. A upper set U is called a Scott open set if for any directed set $A \subseteq D$, $\bigvee A \in U$ implies A intersects U.

[1] Supported by the NSF of China (Nos. 11871353, 12001385) and "the Fundamental Research Funds for the Central Universities" No. 2021SCU12108

[2] Email: zhenchaolyu@scu.edu.cn
[3] Email: xxldannyboy@163.com
[4] Email: kouhui@scu.edu.cn

PUBLISHED MARCH 21, 2023
10.46298/entics.10362
PROCEEDINGS AVAILABLE ONLINE AT
HTTPS://DOI.ORG/10.46298/ENTICS.PROCEEDINGS.ISDT9

For a topological space X, we use $\mathcal{O}(X)$ to denote the lattice of open subsets of X. We require that all topological spaces are T_0 in this note. Let X be a T_0 space, the specializing order $\leq$ is defined as follows : $x \leq y$ if x belongs to the closure of point y. A topological space is a *c-space* if for any $x \in X$ and any open neighbourhood U of x, there is a point $y \in U$ such that $x \in \mathrm{int}\,(\uparrow y)$. A space X is locally finitary compact if for any $x \in X$ and its open neighborhood U, there is a finite subset F of U such that $x \in \mathrm{int}\,(\uparrow F)$.

Let X be a T_0 space and $\leq$ the specialization order over X. A topological space X is called a Scott space if $(X, \leq)$ is a dcpo and the topology on X is equal to the Scott topology on $(X, \leq)$. Every directed set D of X under specialization order can be regarded as a monotone net, we say D converges to x iff for every open neighborhood U of x, $D \bigcap U \neq \emptyset$. We say that V is a directed open set of X if for all directed set D which converges to some point of V, then $D \bigcap V \neq \emptyset$. It is easy to see that every directed open set is an upper set.

Definition 2.1 [12] Let X be a T_0 space. If every directed open set of X is also an open set, then we say that X is a directed space.

There are many important spaces in domain theory which also are directed spaces.

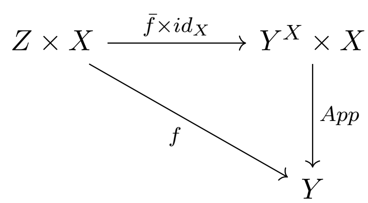
Example 2.2 (i) Every poset with Scott topology is a directed space.

(ii) All c-spaces are directed spaces. In particular, every Alexandroff space is a directed space.

(iii) All locally finitary compact spaces are directed spaces. By the way, every c-space is locally finitary compact.

Next we introduce the concept of the exponential object in general category.

Definition 2.3 Given two objects X, Y in a category $\mathcal{C}$ with binary products, an exponential object, if it exists, is an object Y^X with a morphism $App\colon Y^X \times X \to Y$ such that for every morphism $f\colon Z \times X \to Y$, there is a unique morphism $\bar{f}\colon Z \to Y^X$ such that the following diagram commutes:



The following result describes the underlying set of the exponential object in **Top**.

Proposition 2.4 [9] Let $\mathcal{C}$ be any full subcategory of **Top** with finite products, and assume that $1 = \{\star\}$ is an object of $\mathcal{C}$. Let X, Y be two objects of $\mathcal{C}$ that have an exponential object Y^X in $\mathcal{C}$.

Then there is a unique homeomorphism $\theta\colon Y^X \to [X \to Y]$, for some unique topology on $[X \to Y]$ (the set of all continuous functions from X to Y) , such that $App(h, x) = \theta(h)(x)$ for all $h \in Y^X, x \in X$.

Moreover, $\bar{f}(z)$ is the image by θ^{-1} of $f(z, \_)$ for all $f\colon Z \times X \to Y, z \in Z$.

Remark: By the above result, we always let the exponential object in $\mathcal{C}$ be the set $[X \to Y]$ with some unique topology if it exists.

Theorem 2.5 [12] The category of directed spaces and continuous maps is cartesian closed.

Next, we build a relationship between directed spaces and Scott spaces, which will be used later.

Definition 2.6 Let X be a T_0 space. If X with the specialization order is a dcpo and every open set of X is Scott open in $(X, \leq)$. Then we say that X is a d-space.

Lemma 2.7 *A directed space is a Scott space iff it is a d-space.*

Proof. We only need to show the "if" part. Let X be a d-space and a directed space, obviously every open set of X is Scott open of $(X, \leq)$ since X is a d-space. Now take any Scott open set U of $(X, \leq)$ and for any directed set D converges to some point x of U. Assume that $D \bigcap U = \emptyset$, then $b = \bigvee^{\uparrow} D \notin U$. It follows that $x \in X \backslash {\downarrow} b$. Because D converges to x and $X \backslash {\downarrow} b$ is open in X, there is some $d \in D$ such that $d \in X \backslash {\downarrow} b$,, a contradiction. Hence the assumption is wrong. It means that U is a directed open set of X. Since X is a directed space, the topology on X is exactly the Scott topology on $(X, \leq)$. □

We list some results about separate continuity and joint continuity.

Theorem 2.8 [11] Let E be a T_0 space. The following conditions are equivalent:

(i) E is locally finitary compact.

(ii) For all T_0 space X, if a map from $X \times E$ is separately continuous, then it is jointly continuous.

Corollary 2.9 *Let X be a c-space and Y a T_0 space. For any T_0 space Z, a map $f \colon X \times Y \to Z$ is continuous (i.e. jointly continuous) iff it is separately continuous.*

3 The category of c-spaces

We now prove our main result.

Theorem 3.1 *The category of c-spaces with continuous maps (**CS** for short) is not cartesian closed.*

Proof. Let $\mathbb{Z}^-$ be the set of non-positive integers with the Scott topology. Assume **CS** is a ccc. It is easy to see that the topological product $X \times Y$ is the categorical product because $X \times Y$ is a c-space. Since **CS** is cartesian closed, there exists exponential topology τ on $[\mathbb{Z}^- \to \mathbb{Z}^-]$, we denote by $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$. Then for any c-space Y and any map $f \colon Y \times \mathbb{Z}^- \to \mathbb{Z}^-$, f is continuous iff $\bar{f} \colon Y \to [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is continuous.

Claim 1: The specialization order on $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is equal to the pointwise order.

For any $g_1, g_2 \in [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ with $g_1 \leq_\tau g_2$ $(g_1 \neq g_2)$, take $Y = \mathbb{S}$ with Scott topology. A map $\theta \colon \mathbb{S} \to [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is defined as $\theta(1) = g_2, \theta(0) = g_1$. Then θ is continuous. Hence $\hat{\theta} : \mathbb{S} \times \mathbb{Z}^- \to \mathbb{Z}^-$ is continuous. It follows that

$$g_1(x) = \hat{\theta}(0, x) \leq \hat{\theta}(1, x) = g_2(x)$$

for any $x \in X$.

For any $g_1, g_2 \in [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ with $g_1 \leq g_2$, consider a continuous map $f \colon \mathbb{S} \times \mathbb{Z}^- \to \mathbb{Z}^-$ which is defined as $f(0, x) = g_1(x), f(1, x) = g_2(x)$ $\forall x \in X$. It follows that the transpose map $\bar{f}$ is continuous hence monotone, which implies that

$$g_1 = \bar{f}(0) \leq_\tau \bar{f}(1) = g_2.$$

Claim 2: $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is a d-space.

We only need to show that every directed subfamily $(g_i)_{i \in I}$ of $[\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ converges to its supremum $g = \bigvee^{\uparrow}_{i \in I} g_i$. Let Y be the set $I \cup \{\infty\}$ with the topology generated by $\{\uparrow i \cup \{\infty\} : i \in I\}$. Obviously Y is a c-space. Consider a map $f \colon Y \times \mathbb{Z}^- \to \mathbb{Z}^-$ which is defined as $f(\infty, x) = g(x), f(i, x) = g_i(x)$. It is easy to see that f is continuous since f is continuous iff it is separately continuous by 2.9. It follows that $\bar{f} \colon Y \to [\mathbb{Z}^- \to \mathbb{Z}^-]_\tau$ is continuous, and so $(g_i = \bar{f}(i))_i$ converges to $\bar{f}(\infty) = g$.

Therefore τ is just the Scott topology on $[\mathbb{Z}^- \to \mathbb{Z}^-]$. But from [10] we know that $[\mathbb{Z}^- \to \mathbb{Z}^-]$ is not a continuous domain, hence it is not a c-space, a contradiction. □

Theorem 3.2 [8] A meet continuous dcpo is a continuous dcpo iff it is a quasicontinuous dcpo.

Notice that $[\mathbb{Z}^- \to \mathbb{Z}^-]$ is a meet continuous semilattice which is not continuous, hence it is not a quasicontinuous dcpo. Then we have the following result.

Corollary 3.3 *The category of locally finitary compact spaces with continuous maps is not cartesian closed.*

Acknowledgement

We are grateful to the anonymous referees for useful comments and suggestions. We also would like to thank Yuxu Chen for helpful discussions.

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