

SI_2 -quasicontinuous Spaces¹

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Abstract

In this paper, as a common generalization of SI_2 -continuous spaces and s_2 -quasicontinuous posets, we introduce the concepts of SI_2 -quasicontinuous spaces and \mathcal{GD} -convergence of nets for arbitrary topological spaces by the cuts. Some characterizations of SI_2 -quasicontinuity of spaces are given. The main results are: (1) a space is SI_2 -quasicontinuous if and only if its weakly irreducible topology is hypercontinuous under inclusion order; (2) A T_0 space X is SI_2 -quasicontinuous if and only if the \mathcal{GD} -convergence in X is topological.

Keywords: SI_2 -quasicontinuous poset, weakly irreducible topology, \mathcal{GD} -convergence

1 Introduction

The theory of continuous lattices and domains which arose from computer science and logic, is based on the investigation of directed complete posets (dcpo, for short) in which every directed set has a least upper bound. Since there are important mathematical models where arise posets such as the reals \mathbb{R} and the naturals \mathbb{N} which fail to be dcpo, there is an attempt to carry as much as possible of the theory of continuous lattices to as general an ordered structure as possible (see [14,13,18,15,17,21,23,24]). Moreover, Ern e pointed out the importance of the concept of *standard completions* [4,5,6] in the context of generalized continuous posets. In the absence of enough joins, Ern e introduced the concept of s_2 -continuous posets and the *weak Scott topology* by means of the cuts instead of joins (see [4]), whence they easily apply to generalized settings where certain joins are missing. Quasicontinuous domains introduced by Gierz, Lawson and Stralka [8] capture many of the essential features of continuous domains and pop up from time to time generalizing slightly the powerful notion of continuous domains. Recently they have attracted increased attention through the remarkable work of J. Goubault-Larrecq (see [12]). As a generalization of s_2 -continuity, the concept of s_2 -quasicontinuity was introduced in [24] by Zhang and Xu, their basic idea is to generalize the way-below relation between the points to the case of sets. It was proved that s_2 -quasicontinuous posets equipped with the weak Scott topologies are precisely the hypercontinuous lattices. It is well known that given a topological space one can also define ordered structures (see [5,11,7,25]). At the

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6th International Symposium in Domain Theory, J.D. Lawson emphasized the need to develop the core of domain theory directly in T_0 topological spaces instead of posets. Moreover, it was pointed out that several results in domain theory can be lifted from the context of posets to T_0 topological spaces (see [7,25,2,3]). In [7], Ern e further proved that the weak Scott topology is the weakest monotone determined topology with a given specialization order. In [11], the concept of \mathcal{S} -convergence for dcpos was introduced by Scott to characterize continuous domains. It was proved that for a dcpo, the \mathcal{S} -convergence is topological if and only if it is a continuous domain. The strong interplay between order theory and topology is often exhibited by the use of different types of convergence; in particular, various notions of continuity of ordered structures can be characterized in this way. In [25], Zhao and Ho defined a new way-below relation and a new topology constructed from any given topology on a set using irreducible sets in a T_0 topological space replacing directed subsets and investigated the properties of this derived topology and k -bounded spaces. It was proved that a space X is SI -continuous if and only if $SI(X)$ is a C -space.

In this paper, we introduce the concepts of SI_2 -quasicontinuous spaces and \mathcal{GD} -convergence of nets for arbitrary topological spaces by the cuts. Some characterizations of SI_2 -quasicontinuity of spaces are given. We show that a space is SI_2 -quasicontinuous if and only if its weakly irreducible topology is hypercontinuous under inclusion order. Finally we arrive at the conclusion that a T_0 space X is SI_2 -quasicontinuous if and only if the \mathcal{GD} -convergence in X is topological.

2 Preliminaries

Let P be a partially ordered set (poset, for short). A nonempty set $D \subseteq P$ is *directed* if for any d_1, d_2 in D there exists d in D above d_1 and d_2 . For $A \subseteq P$, $\uparrow A = \{x \in P : a \leq x \text{ for some } a \in A\}$. A subset A of P is an *upper set* if $A = \uparrow A$. The order dual concepts are $\downarrow A$ and *lower set*. A^\uparrow and A^\downarrow denote the sets of all *upper* and *lower bounds* of A , respectively. A *cut* δ of A in P is defined by $A^\delta = (A^\uparrow)^\downarrow$. We say that a nonempty family \mathcal{G} of sets is *directed* if given G_1, G_2 in the family \mathcal{G} , there exists $G \in \mathcal{G}$ such that $G_1, G_2 \leq G$, i.e., $G \subseteq \uparrow G_1 \cap \uparrow G_2$. For nonempty subsets F and G of P , we say F *approximates* G if for every directed subset $D \subseteq P$, whenever $\bigvee D$ exists, $\bigvee D \in \uparrow G$ implies $d \in \uparrow F$ for some $d \in D$. A dcpo P is called a *quasicontinuous domain* if for all $x \in P$, $\uparrow x$ is the directed (with respect to reverse inclusion) intersection of sets of the form $\uparrow F$, where F approximates $\{x\}$ and F is a finite set of P . Let $P^{(<\omega)}$ be the set of all nonempty finite subsets of P .

Given a poset P , we can generate some intrinsic topologies. The upper sets form the *Alexandroff topology* $\alpha(P)$. For all directed sets D of P , the *weak Scott topology* $\sigma_2(P)$ consists of all upper sets U such that $D^\delta \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$. Obviously if P is a dcpo, then the weak Scott topology coincides with the usual *Scott topology* $\sigma(P)$, which consists of all upper sets U such that $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ for all directed sets D in P . For $x \in P$, the topology generated by the collection of sets $P \setminus \downarrow x$ (as a subbase) is called the *upper topology* and is denoted by $\nu(P)$. It is easy to see that $\nu(P) \subseteq \sigma_2(P) \subseteq \sigma(P) \subseteq \alpha(P)$.

For a topological space (X, τ) , the *specialization order* \leq on X is defined by $y \leq x$ if and only if $y \in \text{cl}(\{x\})$. If (X, τ) is T_0 then the specialization order is a partial order. The specialization order on X is denoted by \leq_τ if there is need to emphasize the topology τ . Keep in mind that the specialization order of the Alexandroff topology on a poset coincides with the underlying order.

Let (X, τ) be a topological space. A nonempty subset $F \subseteq X$ is called *irreducible* if for every closed sets B and C , whenever $F \subseteq B \cup C$, one has either $F \subseteq B$ or $F \subseteq C$. The set of all irreducible sets of the topological space (X, τ) will be denoted by $\text{Irr}_\tau(X)$ or $\text{Irr}(X)$.

Unless otherwise stated, in the context of T_0 spaces, all order-theoretic concepts such as lower sets, upper sets, etc. are taken with respect to the specialization order of the underlying spaces.

Lemma 2.1 *Let (X, τ) be a T_0 space.*

- (1) *If $D \subseteq X$ is a directed set with respect to the specialization order, then D is irreducible;*
- (2) *If $U \subseteq X$ is an open set, then U is an upper set; Similarly, if $F \subseteq X$ is a closed set, then F is a lower set.*

Definition 2.2 ([4,24]) *Let P be a poset and $x \in P$, $A, B \subseteq P$.*

- (1) We say that A is *way below* B and write $A \ll B$ if for all directed sets $D \subseteq P$, $B \cap D^\delta \neq \emptyset$ implies $A \cap D \neq \emptyset$. We write $A \ll x$ for $A \ll \{x\}$ and $y \ll B$ for $\{y\} \ll B$. Let $w(x) = \{F \in P^{(<\omega)} : F \ll x\}$, $\Downarrow x = \{y \in P : y \ll x\}$.
- (2) P is called *s_2 -continuous* if for all $x \in P$, $x \in (\Downarrow x)^\delta$ and $\Downarrow x$ is directed.
- (3) P is called *s_2 -quasicontinuous* if for each $x \in P$, $\uparrow x = \bigcap \{\uparrow F : F \text{ is finite, } F \ll x\}$ and $w(x) = \{\uparrow F \subseteq P : F \text{ is finite and } F \ll x\}$ is directed.

Definition 2.3 ([25]) Let (X, τ) be a T_0 space. For $x, y \in X$, define $x \ll_{SI} y$ if for all irreducible sets F , $y \leq \bigvee F$ implies there exists $e \in F$ such that $x \leq e$ whenever $\bigvee F$ exists. The set $\{y \in X : y \ll_{SI} x\}$ is denoted by $\Downarrow_{SI} x$ and the set $\{y \in X : x \ll_{SI} y\}$ by $\Uparrow_{SI} x$.

Definition 2.4 ([16,20]) Let (X, τ) be a T_0 space and $x, y \in X$. Define $x \ll_r y$ if for every irreducible set E , $y \in E^\delta$ implies there exists $e \in E$ such that $x \leq e$. We denote the set $\{y \in X : y \ll_r x\}$ by $\Downarrow_r x$ and the set $\{y \in X : x \ll_r y\}$ by $\Uparrow_r x$.

Definition 2.5 ([20]) Let (X, τ) be a T_0 space. X is called *SI_2 -continuous* if the following conditions are satisfied:

- (1) $\Downarrow_r x$ is directed for all $x \in X$;
- (2) $x \in (\Downarrow_r x)^\delta$ for all $x \in X$;
- (3) $\Uparrow_r x \in \tau$ for all $x \in X$.

Definition 2.6 ([16,20]) Let (X, τ) be a T_0 space. A subset $U \subseteq X$ is called *weakly irreducibly open* if the following conditions are satisfied:

- (1) $U \in \tau$;
- (2) $F^\delta \cap U \neq \emptyset$ implies $F \cap U \neq \emptyset$ for all $F \in \text{Irr}_\tau(X)$.

It is easy to check that the set of all weakly irreducibly open sets of (X, τ) is a topology, which will be called *weakly irreducible topology* of X and will be denoted by $\tau_{SI_2}(X)$. $SI_2(X, \tau) = (X, \tau_{SI_2}(X))$ denotes the weakly irreducible topological space.

The following lemma is the well-known Rudin Lemma.

Lemma 2.7 ([11]) *Let \mathcal{F} be a directed family of nonempty finite subsets of a poset P . Then there exists a directed set $D \subseteq \bigcup_{F \in \mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

Definition 2.8 ([20]) Let (X, τ) be a T_0 space. A net $(x_j)_{j \in J}$ in X is said to converge to $x \in X$ if there exists a directed set $D \subseteq X$ with respect to the specialization order such that

- (1) $x \in D^\delta$;
- (2) For all $d \in D$, $d \leq x_j$ holds eventually.

In this case we write $x \equiv_{\mathcal{D}} \lim x_j$.

3 SI_2 -quasicontinuous spaces

In this section, the concept of SI_2 -quasicontinuous spaces is introduced. Some properties of SI_2 -quasicontinuous spaces are discussed.

Definition 3.1 ([16]) Let X be a T_0 space and $x \in X$, $A, B \subseteq X$. We say that A is *way below* B and write $A \ll_r B$ if for all irreducible sets $E \subseteq X$, $B \cap E^\delta \neq \emptyset$ implies $A \cap E \neq \emptyset$. We write $A \ll_r x$ for $A \ll_r \{x\}$ and $y \ll_r B$ for $\{y\} \ll_r B$. The set $\{x \in X : A \ll_r x\}$ is denoted by $\Uparrow_r A$.

Definition 3.2 A T_0 space (X, τ) is called *SI_2 -quasicontinuous* if for all $x \in X$, the following conditions are satisfied:

- (1) $w(x) = \{F \subseteq X : F \in X^{(<\omega)} \text{ and } F \ll_r x\}$ is directed;
- (2) $\uparrow x = \bigcap \{\uparrow F : F \in w(x)\}$;
- (3) For any $H \in X^{(<\omega)}$, $\uparrow_r H \in \tau$.

The following proposition is simple and the proof is omitted.

Proposition 3.3 *Let X be a T_0 space and $G, H, K, M \subseteq X$. Then*

- (1) $G \ll_r H \Leftrightarrow G \ll_r h$ for all $h \in H$;
- (2) $G \ll_r H \Leftrightarrow \uparrow G \ll_r \uparrow H$;
- (3) $G \ll_r H \Rightarrow G \leq H$, that is, $\uparrow H \subseteq \uparrow G$. Hence $\uparrow H \subseteq \bigcap \{\uparrow F : F \in w(H)\}$.
- (4) $G \leq H \ll_r K \leq M \Rightarrow G \ll_r M$.

Clearly, an SI_2 -continuous space must be SI_2 -quasicontinuous, but the converse may not be true.

Example 3.4 Let $P = \{a\} \cup \{a_n : n \in \mathbb{N}\}$, where \mathbb{N} denotes the set of all positive integers. The partial order on P is defined by setting $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and $a_1 \leq a$. Then P endowed with the Alexandroff upper topology is an SI_2 -quasicontinuous space which is not SI_2 -continuous.

Lemma 3.5 *Let P be a poset. Then $SI_2(P, \alpha(P)) = (P, \sigma_2(P))$.*

- Remark 3.6** (1) Let P be a poset. Then P is an s_2 -quasicontinuous poset if and only if it is an SI_2 -quasicontinuous space with respect to the Alexandroff topology.
- (2) Let (X, τ) be a T_0 space. If X is an SI_2 -quasicontinuous space, then it is also an s_2 -quasicontinuous poset under the specialization order. But the converse may not be true.

Example 3.7 Let X be an infinite set with a cofinite topology τ . Then it is a T_1 space. Clearly it is an antichain under the specialization order, and hence it is an s_2 -continuous poset. Thus it is also s_2 -quasicontinuous. But $\uparrow_r x = \{x\} \notin \tau$ for all $x \in X$, then (X, τ) is not an SI_2 -quasicontinuous space.

Lemma 3.8 ([16]) *Let \mathcal{F} be a directed family of nonempty finite sets in a T_0 space X . If $G \ll_r x$ with $\bigcap \mathcal{F} \subseteq \uparrow x$ then there exists some $F \in \mathcal{F}$ such that $F \subseteq \uparrow G$.*

We now derive the interpolation property of SI_2 -quasicontinuous spaces.

Theorem 3.9 *Let X be an SI_2 -quasicontinuous space. Then*

- (1) For $x \in X, F \ll_r x$ implies that there exists some $G \in w(x)$ such that $F \ll_r G \ll_r x$.
- (2) For any $F, H \in X^{(<\omega)}$, $F \ll_r H$ implies that there exists some $G \in X^{(<\omega)}$ such that $F \ll_r G \ll_r H$.
- (3) For all $F \in X^{(<\omega)}, E \in Irr(X), F \ll_r E^\delta$ implies that there exists some $e \in E$ with $F \ll_r e$.

Proof. (1) The statement has been proved for quasicontinuous domains in [3, Proposition III-3.5], and the similar proof carries over to this setting.

(2) For all $H \in P^{(<\omega)}, F \subseteq X$, if $F \ll_r H$, then for all $h \in H$, $F \ll_r h$ by Proposition 3.3. By (1), there exists a finite set G_h such that $F \ll_r G_h \ll_r h$. Let $G = \bigcup_{h \in H} G_h$. Then $G \in P^{(<\omega)}$ and $F \ll_r G \ll_r H$.

(3) For all $F \in X^{(<\omega)}, E \in Irr(X)$, if $F \ll_r E^\delta$, then there exist $G, H \in X^{(<\omega)}$ such that $F \ll_r G \ll_r H \ll_r E^\delta$. Hence there are $g \in G, e \in E$ with $g \leq e$. Clearly, $F \ll_r g$, and then $F \ll_r e$. \square

Proposition 3.10 *Let (X, τ) be an SI_2 -quasicontinuous space.*

- (1) For any $H \in X^{(<\omega)}$, $\uparrow_r H = \text{int}_{\tau_{SI_2(X)}} \uparrow H$.
- (2) A subset U of X is weakly irreducibly open iff for each $x \in U$ there exists a finite $F \ll_r x$ such that $\uparrow F \subseteq U$.
- (3) The sets $\{\uparrow_r F : F \in X^{(<\omega)}\}$ form a basis for $\tau_{SI_2(X)}$.

Proof. (1) Suppose that H is a nonempty set in X . It is easy to show that $\text{int}_{\tau_{SI_2}(X)} \uparrow H \subseteq \uparrow_r H$. Conversely, we show that $\uparrow_r H \in \tau_{SI_2}(X)$. Obviously, $\uparrow_r H \in \tau$. For all irreducible sets $E \subseteq X$, if $E^\delta \cap \uparrow_r H \neq \emptyset$, i.e., there exists $e \in E^\delta \cap \uparrow_r H$, then there exists a $F \in X^{(<\omega)}$ such that $H \ll_r F \ll_r e$ by Theorem 3.9. Since $e \in E^\delta$, $\uparrow F \cap E \neq \emptyset$, i.e., there exists some $y \in \uparrow F \cap E$. Thus $y \in \uparrow_r H \cap E \neq \emptyset$ by Proposition 3.3. So $\uparrow_r H \in \tau_{SI_2}(X)$. Since $\uparrow_r H \subseteq \uparrow H$, $\uparrow_r H \subseteq \text{int}_{\tau_{SI_2}(X)} \uparrow H$.

(2) Let $U \in \tau_{SI_2}(X)$, $x \in U$. By the definition of the weakly irreducibly open set, we have $U \ll_r x$, so by Theorem 3.9, there exists $F \in X^{(<\omega)}$ such that $U \ll_r F \ll_r x$. Thus $\uparrow F \subseteq U$. Conversely, suppose that the condition is satisfied, we have $U = \cup \{\uparrow_r F : F \in X^{(<\omega)}, \uparrow F \subseteq U\}$. Since the space X is SI_2 -quasicontinuous, so $U \in \tau$. For all irreducible sets $E \subseteq X$, if $U \cap E^\delta \neq \emptyset$, then there exists $y \in U \cap E^\delta$. By hypothesis, there exists a finite $G \ll_r y$ such that $\uparrow G \subseteq U$. Therefore, $E \cap U \supseteq E \cap \uparrow G \neq \emptyset$. Thus U is weakly irreducibly open.

(3) Let $U \in \tau_{SI_2}(X)$. By (2), we have $U = \cup \{\uparrow_r F : F \in X^{(<\omega)}, \uparrow F \subseteq U\}$. By (1), $\uparrow_r F \in \tau_{SI_2}(X)$ for all $F \in X^{(<\omega)}$. Thus the sets $\{\uparrow_r F : F \in X^{(<\omega)}\}$ form a basis for $\tau_{SI_2}(X)$. \square

Definition 3.11 ([10]) For a complete lattice L , define a relation \prec on L by $x \prec y \Leftrightarrow y \in \text{int}_{v(L)} \uparrow x$. L is called hypercontinuous if $x = \bigvee \{u \in L : u \prec x\}$ for all $x \in L$.

Now we give the topological characterizations of SI_2 -quasicontinuous spaces.

Theorem 3.12 For a T_0 space X , the following statements are equivalent:

- (1) X is an SI_2 -quasicontinuous space;
- (2) For all $x \in X$ and $U \in \tau_{SI_2}(X)$, $x \in U$ implies that there exists $F \in X^{(<\omega)}$ such that $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow F \subseteq \uparrow F \subseteq U$;
- (3) $(\tau_{SI_2}(X), \subseteq)$ is a hypercontinuous lattice.

Proof. (1) \Rightarrow (2): It follows from Proposition 3.10.

(2) \Rightarrow (1): For all $x \in P$, let $\mathcal{F} = \{F \in P^{(<\omega)} : x \in \text{int}_{\tau_{SI_2}(X)} \uparrow F\}$. Since $X \in \tau_{SI_2}(X)$, it follows from (2) that there exists $F \in X^{(<\omega)}$ such that $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow F \subseteq \uparrow F \subseteq X$. Then $F \in \mathcal{F} \neq \emptyset$. Obviously $\uparrow x \subseteq \bigcap_{F \in \mathcal{F}} \uparrow F$. If $x \not\leq y$, then $x \in X \setminus \downarrow y \in \tau_{SI_2}(X)$. By (2), there exists $F \in X^{(<\omega)}$ such that $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow F \subseteq \uparrow F \subseteq X \setminus \downarrow y$. Then $F \in \mathcal{F}$ and $y \notin \uparrow F$. Thus $\uparrow x = \bigcap_{F \in \mathcal{F}} \uparrow F$. Now we show that \mathcal{F} is directed. Let $F_1, F_2 \in \mathcal{F}$, we have $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow F_1 \cap \text{int}_{\tau_{SI_2}(X)} \uparrow F_2 \in \tau_{SI_2}(X)$. Again from (2), there exists $F \in X^{(<\omega)}$ such that $x \in \text{int}_{\tau_{SI_2}(X)} \uparrow F \subseteq \uparrow F \subseteq \text{int}_{\tau_{SI_2}(X)} \uparrow F_1 \cap \text{int}_{\tau_{SI_2}(X)} \uparrow F_2 \subseteq \uparrow F_1 \cap \uparrow F_2$. Thus we have that \mathcal{F} is directed and $F \in \mathcal{F}$. By the definition of $\tau_{SI_2}(X)$, $F \ll_r x$ for all $F \in \mathcal{F}$, and then $\mathcal{F} \subseteq w(x)$. By Lemma 3.8, it is not difficult to show that $w(x)$ is directed. So $\uparrow x \subseteq \bigcap_{F \in w(x)} \uparrow F \subseteq \bigcap_{F \in \mathcal{F}} \uparrow F = \uparrow x$.

(2) \Leftrightarrow (3): It follows from Lemma 2.2 of [22]. \square

4 \mathcal{GD} -convergence in SI_2 -quasicontinuous spaces

In this section, the concept of \mathcal{GD} -convergence in a poset is introduced. It is proved that the T_0 space X is SI_2 -quasicontinuous if and only if the \mathcal{GD} -convergence in X is topological.

Definition 4.1 Let X be a T_0 space and $(x_j)_{j \in J}$ a net in X . $F \subseteq X$ is called a quasi-eventual lower bound of a net $(x_j)_{j \in J}$ in P , if F is finite and there exists $k \in J$ such that $x_j \in \uparrow F$ for all $j \geq k$ with respect to the specialization order.

Definition 4.2 Let X be a T_0 space and $(x_j)_{j \in J}$ a net. We say $x \in X$ is a \mathcal{GD} -limit of the net $(x_j)_{j \in J}$ if there exists a directed family $\mathcal{F} = \{F \subseteq X : F \text{ is finite}\}$ of quasi-eventual lower bounds of the net $(x_j)_{j \in J}$

in X with respect to the specialization order such that $\bigcap_{F \in \mathcal{F}} \uparrow F \subseteq \uparrow x$. In this case we write $x \equiv_{\mathcal{GD}} \lim x_j$.

Lemma 4.3 *Let X be a T_0 space and $(x_j)_{j \in J}$ a net in X . If $x \equiv_{\mathcal{D}} \lim x_j$ then $x \equiv_{\mathcal{GD}} \lim x_j$.*

Proof. Let X be a T_0 space and $(x_i)_{i \in I}$ a net with $x \equiv_{\mathcal{D}} \lim x_j$. Then there is a directed set D of an eventual lower bounds of a net $(x_j)_{j \in J}$ with $x \in D^\delta$. Let $\mathcal{F} = \{\{d\} : d \in D\}$, then \mathcal{F} is a directed family of a quasi-eventual lower bounds of a net $(x_j)_{j \in J}$ and $D^\uparrow = \bigcap \{\uparrow d : d \in D\} \subseteq \uparrow x$. Thus $x \equiv_{\mathcal{GD}} \lim x_j$. \square

Remark 4.4 Let X be a T_0 space and $(x_j)_{j \in J}$ a net in X . If $x \equiv_{\mathcal{GD}} \lim x_j$ then we may not have $x \equiv_{\mathcal{D}} \lim x_j$.

Example 4.5 Let $P = \mathbb{N} \cup \{\top, z\}$, where \mathbb{N} denotes the set of all natural numbers. The order \leq on P is defined as follows (see Figure 1):

- (1) $\forall x \in P, x \leq \top$;
- (2) $\forall x, y \in \mathbb{N}, x \leq y$ if x is less than or equal to y according to the usual order on natural numbers.

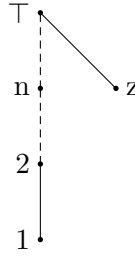


Fig. 1. A \mathcal{GD} -limit of a net for which is not a \mathcal{D} -limit.

Then $(P, \alpha(P))$ is SI_2 -quasicontinuous but not SI_2 -continuous. Also for all $n \in \mathbb{N}, \{z, n\} \ll z$ and $\uparrow z = \bigcap_{n \in \mathbb{N}} \uparrow \{z, n\}$. Let $x_{2n} = n, x_{2n+1} = z$, then $(x_j)_{j \in \mathbb{N}}$ is a net and $\{z, n\}$ is a quasi-eventual lower bound of it. Hence $z \equiv_{\mathcal{GD}} \lim x_n$. It is not difficult to check that $z \leq x_n$ does not hold eventually. Thus z is not a \mathcal{D} -limit of the net $(x_n)_{n \in \mathbb{N}}$.

Proposition 4.6 *Let \mathcal{F} be a directed family of nonempty finite sets in a T_0 space X . If $x \in U \in \tau_{SI_2}(X)$ and $\bigcap_{F \in \mathcal{F}} \uparrow F \subseteq \uparrow x$, then $F \subseteq U$ for some $F \in \mathcal{F}$.*

Proof. Suppose not, then the collection $\{F \setminus U : F \in \mathcal{F}\}$ is a directed family of nonempty finite sets. By Lemma 2.7, there is some directed set $D \subseteq \bigcup \{F \setminus U : F \in \mathcal{F}\}$ such that $D \cap (F \setminus U) \neq \emptyset$ for all $F \in \mathcal{F}$. Then $D^\uparrow = \bigcap_{d \in D} \uparrow d \subseteq \bigcap_{F \in \mathcal{F}} \uparrow (F \setminus U) \subseteq \bigcap_{F \in \mathcal{F}} \uparrow F \subseteq \uparrow x$. Thus $x \in (D^\uparrow)^\downarrow = D^\delta$. Now we have $x \in D^\delta \cap U \neq \emptyset$, and hence $D \cap U \neq \emptyset$ by the definition of the weakly irreducibly open set. This implies $d \in U$ for some $d \in D$, which contradicts the fact that d belongs to some $F \setminus U$ which is disjoint from U . \square

Now we derive the \mathcal{GD} -convergence in the SI_2 -quasicontinuous space is topological.

Proposition 4.7 *Let X be an SI_2 -quasicontinuous space. Then $x \equiv_{\mathcal{GD}} \lim x_j$ if and only if the net $(x_j)_{j \in J}$ converges to the element x with respect to the topology $\tau_{SI_2}(X)$. That is, the \mathcal{GD} -convergence is topological.*

Proof. Let X be SI_2 -quasicontinuous. Suppose first that $x \equiv_{\mathcal{GD}} \lim x_j$ and $x \in U \in \tau_{SI_2}(X)$. Then there exists a directed family $\mathcal{F} = \{F \subseteq X : F \text{ is finite}\}$ of a quasi-eventual lower bounds of the net $(x_j)_{j \in J}$ in X with respect to the specialization order such that $\bigcap_{F \in \mathcal{F}} \uparrow F \subseteq \uparrow x$. By Proposition 4.6, there exists some $F \in \mathcal{F}$ such that $F \subseteq U$. Thus there exists $j_0 \in J$ such that $x_j \in \uparrow F \subseteq U$ for all $j \geq j_0$. This shows that the net $(x_j)_{j \in J}$ converges to the element x with respect to the topology $\tau_{SI_2}(X)$. Conversely, we suppose that the net $(x_j)_{j \in J}$ converges to an element x with respect to the topology $\tau_{SI_2}(X)$. Since X is SI_2 -quasicontinuous, there exists a directed family $w(x) = \{F \subseteq X : F \in X^{(\omega)} \text{ and } F \ll_r x\}$ and

$\uparrow x = \bigcap \{\uparrow F : F \in w(x)\}$. For all $F \in w(x)$, let $U_F = \{y \in X : F \ll_r y\}$. Then $x \in U_F$ and $U_F \in \tau_{SI_2}(X)$ by Proposition 3.10, and hence $x_j \in U_F$ eventually holds, that is to say, there exists $j_0 \in J$ such that $F \ll_r x_j$ for all $j \geq j_0$ which implies $x_j \in \uparrow F$. Moreover since $w(x)$ is directed, and then one has $x \equiv_{\mathcal{GD}} \lim x_j$. \square

The converse is also true.

Proposition 4.8 *Let (X, τ) be a T_0 space. If the \mathcal{GD} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological, then X is SI_2 -quasicontinuous.*

Proof. Suppose that the \mathcal{GD} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological. For any $x \in X$, let $J = \{(U, n, a) \in N(x) \times \mathbb{N} \times X : a \in U\}$, where $N(x)$ consists of all weakly irreducibly open sets which contain x , and define an order on J as follows: $(U, m, a) \leq (V, n, b)$ if and only if V is proper subset of U or $U = V$ and $m \leq n$. Obviously, J is directed. Let $x_j = a$ for all $j = (U, m, a) \in J$. Then it is not difficult to check that the net $(x_j)_{j \in J}$ converges to the element x with respect to the weakly irreducible topology $\tau_{SI_2}(X)$, and hence $x \equiv_{\mathcal{GD}} \lim x_j$. Thus it concludes that there is a directed family $\mathcal{F} = \{F \subseteq X : F \text{ is finite}\}$ of a quasi-eventual lower bounds of the net $(x_j)_{j \in J}$ in X with respect to the specialization order such that $\bigcap_{F \in \mathcal{F}} \uparrow F \subseteq \uparrow x$. Now we prove that (1) for all $F \in \mathcal{F}, F \ll_r x$; (2) $\bigcap_{F \in \mathcal{F}} \uparrow F = \uparrow x$.

(1) Let $E \subseteq X$ be irreducible with $x \in E^\delta$. Since F is a quasi-eventual lower bound of the net $(x_j)_{j \in J}$, there is $j_0 = (U, m, a) \in J$ such that $x_j \in \uparrow F$ for all $j = (V, n, b) \geq j_0$. Notice $x \in U$, so $E^\delta \cap U \neq \emptyset$, and then $E \cap U \neq \emptyset$. Pick $e \in E \cap U$. Set $i = (U, m+1, e)$, then $i \geq (U, m, a) = j_0$. Thus $e = x_i \in \uparrow F$, that is, $F \ll_r x$.

(2) We only need to show that $\uparrow x \subseteq \bigcap_{F \in \mathcal{F}} \uparrow F$. Suppose not, then there exists $y \geq x$ but $y \notin \bigcap_{F \in \mathcal{F}} \uparrow F$, that is, there exists $F \in \mathcal{F}$ with $y \notin \uparrow F$. And then $\uparrow F \subseteq X \setminus \downarrow x$. Again since F is a quasi-eventual lower bound of the net $(x_j)_{j \in J}$, there exists $j_0 = (U, m, a) \in J$ such that $x_j \in \uparrow F$ for all $j = (V, n, b) \geq j_0$. Now we have $x \in U$. Set $i = (U, m+1, x)$, then $i \geq (U, m, a) = j_0$. Thus $x = x_i \in \uparrow F \subseteq X \setminus \downarrow x$, a contradiction. Finally, it claims that $\uparrow_r H \in \tau$ for all nonempty finite sets $H \subseteq X$. Indeed, if $y \in \uparrow_r H$, then there exists a nonempty finite set $K \subseteq X$ such that $H \ll_r K \ll_r y$ by Theorem 3.9. From the above argument, as long as we replace $J = \{(U, n, a) \in N(x) \times \mathbb{N} \times X : a \in U\}$ with $I = \{(U, n, a) \in N(y) \times \mathbb{N} \times X : a \in U\}$, where $N(y)$ consists of all weakly irreducibly open sets containing y , similarly we have that $(x_i)_{i \in I}$ converges to y with respect to the topology $\tau_{SI_2}(X)$, and then there exists directed family $\mathcal{G} = \{G \subseteq X : G \text{ is finite}\}$ of a quasi-eventual lower bounds of the net $(x_j)_{j \in J}$ in X with respect to the specialization order such that $\bigcap_{G \in \mathcal{G}} \uparrow G \subseteq \uparrow y$. Note that $K \ll_r y$. By Lemma 3.8, there exists $G \in \mathcal{G}$ such that $G \subseteq \uparrow K$. By the convergence of the net, there exists $i_0 = (U_1, m, a)$ such that $x_i = b \in \uparrow G$ for all $i \geq i_0$, where $U_1 \in \tau_{SI_2}(X)$. In particular, we have $(U_1, m+1, u) \geq (U_1, m, a) = i_0$ for all $u \in U_1$. We conclude that $u \in \uparrow G$, and hence $U_1 \subseteq \uparrow G$. Next we show that $U_1 \subseteq \uparrow_r H$. Let $u \in U_1$ and $E \in \text{Irr}(X)$ with $u \in E^\delta \cap U_1$. Since U_1 is weakly irreducibly open, then $E \cap U_1 \neq \emptyset$. Pick $e \in E \cap U_1$. And then we have $e \in E \cap U_1 \subseteq E \cap \uparrow G \subseteq E \cap \uparrow K \subseteq E \cap \uparrow H$. Thus we have $H \ll_r u$, and hence we have $y \in U_1 \subseteq \uparrow_r H$. This shows that $\uparrow_r H \in \tau$. Hence X is SI_2 -quasicontinuous. \square

From Proposition 4.7 and Proposition 4.8 we have:

Theorem 4.9 *Let X be a T_0 space. Then the following statements are equivalent:*

- (1) SI_2 -quasicontinuous space;
- (2) The \mathcal{GD} -convergence with respect to the topology $\tau_{SI_2}(X)$ is topological, that is, for all $x \in X$ and all nets $(x_j)_{j \in J}$ in X , $x \equiv_{\mathcal{GD}} \lim x_j$ if and only if $(x_j)_{j \in J}$ converges to x with respect to the weakly irreducible topology.

Corollary 4.10 ([26]) *Let P be a dcpo. Then the following conditions are equivalent:*

- (1) P is a quasicontinuous domain;
- (2) S^* -convergence in P is topological for the Scott topology $\sigma(P)$, that is, for all $x \in P$ and all nets $(x_j)_{j \in J}$ in P , $(x_j)_{j \in J}$ S^* -converges to x if and only if $(x_j)_{j \in J}$ converges to x with respect to the Scott

topology.

Corollary 4.11 ([19]) *Let P be a poset. Then the following conditions are equivalent:*

- (1) P is s_2 -quasicontinuous;
- (2) The \mathcal{GS} -convergence in P is topological for the weak Scott topology $\sigma_2(P)$, that is, for all $x \in P$ and all nets $(x_j)_{j \in J}$ in P , $x \equiv_{\mathcal{GS}} \lim x_j$ if and only if $(x_j)_{j \in J}$ converges to x with respect to the weak Scott topology.

5 Conclusions

In this paper, we introduce the concepts of SI_2 -quasicontinuous spaces and present one way to generalize \mathcal{D} -convergence of nets for arbitrary topological spaces by the cuts. Some convergence theoretical characterizations of SI_2 -quasicontinuity of spaces are given and come to the main conclusions are: (1) A space is SI_2 -quasicontinuous if and only if its weakly irreducible topology is hypercontinuous under inclusion order; (2) A T_0 space X is SI_2 -quasicontinuous if and only if the \mathcal{GD} -convergence in X is topological.

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