Electronic Notes in
Theoretical Informatics
And Computer Science

ENTICS
https://entics.episciences.org

Volume 2
Proceedings of
ISDT 2022

One-step Closure, Weak One-step Closure and Meet Continuity

Hualin Miao[a,1,2] Qingguo Li[a,1,3] Dongsheng Zhao[b,4]

[a] *School of Mathematics*
Hunan University
Changsha, Hunan, 410082, China

[b] *School of Mathematics*
Nanyang Technological University
1 Nanyang Walk, Singapore 637616

Abstract

This paper studies the weak one-step closure and one-step closure properties concerning the structure of Scott closures. We deduce that every quasicontinuous domain has weak one-step closure and show that a quasicontinuous poset need not have weak one-step closure. We also constructed a non-continuous poset with one-step closure, which gives a negative answer to an open problem posed by Zou et al.. Finally, we investigate the relationship between weak one-step closure property and one-step closure property and prove that a poset has one-step closure if and only if it is meet continuous and has weak one-step closure.

Keywords: Weak one-step closure, One-step closure, Quasicontinuous domain, Quasicontinuous poset, Continuous poset

1 Introduction

The Scott topology is an intrinsic topology on posets, which is the most important topology in domain theory. Scott proved that a domain endowed with the Scott topology is sober. It is well known that a poset is continuous if and only if its Scott closed set lattice is a completely distributive lattice. In [7], Zhao introduced the weak one-step closure property in order to obtain some characterizations of Z-continuous posets. In [6], Zou et al. proposed the one-step closure property and proved that every continuous poset has one-step closure. They asked whether all posets with one-step closure are continuous. Since every continuous poset is quasicontinuous, it is natural to wonder whether every quasicontinuous poset also has one-step closure.

In this paper we shall answer the above problems and investigate other aspects of weak one-step closure and one-step closure properties. We give the outline of this paper below.

[1] This work is supported by the National Natural Science Foundation of China (No.12231007) and by Hunan Provincial Innovation Foundation For Postgraduate (CX20200419)

[2] Email: miaohualinmiao@163.com

[3] Corresponding author, Email: liqingguoli@aliyun.com

[4] Email: dongsheng.zhao@nie.edu.sg

Published March 20, 2023
10.46298/entics.10352

Proceedings Available Online at
https://doi.org/10.46298/entics.proceedings.isdt9

In Section 3, we prove that every quasicontinuous domain has weak one-step closure and show, by a counterexample, that a quasicontinuous poset may not have weak one-step closure. In Section 4, we give a negative answer to the problem posed by Zou et al. in [6]. In Section 5, we prove that a poset has one-step closure if and only if it is meet continuous and has weak one-step closure.

Some problems are posed for further investigation.

2 Preliminaries

We now recall some basic notions and results to be used later. We refer the readers to [3], [2] for more about these.

Let P be a poset. For any subset A of P, let $\uparrow A = \{y \in P : x \leq y \text{ for some } x \in A\}$ and $\downarrow A = \{y \in P : y \leq x \text{ for some } x \in A\}$. A nonempty subset D of P is *directed*, denoted by $D \subseteq^{\uparrow} P$, if every finite subset of D has an upper bound in D. The *supremum (infimum)* of a subset A of P, if exists, means the least upper (greatest lower) bound of A in P and will be denoted by $\sup A$ ($\inf A$, resp.) A *semilattice* is a poset in which every nonempty finite subset has an inf; the dual notion is the *sup semilattice*. A *Scott open* subset of P is an upper set U ($U = \uparrow U$) of P such that, for every directed subset D of P such that $\sup D$ exists and is in U, there is a $d \in D$ such that $d \in U$. The complements of Scott open sets are called *Scott closed sets*. The collection of all Scott open subsets of P form a topology on P, which is called the *Scott topology* of P and denoted by $\sigma(P)$. The collection of all Scott closed subsets of P is denoted by $\Gamma(P)$. The space $(P, \sigma(P))$ is simply written as ΣP. For any $A \subseteq P$, we write $cl(A)$ as the Scott closure of A (the closure of A with respect to the Scott topology). We denote the set of all finite subsets of a poset P by $Fin(P)$. The Smyth preoder on the set of all subsets of P is given by $G \leq H$ if $\uparrow H \subseteq \uparrow G$. We say that G is way below H and write $G \ll H$ if for every directed subset $D \subseteq P$, $\sup D \in \uparrow H$ implies $D \cap \uparrow G \neq \emptyset$. We write $G \ll x$ for $G \ll \{x\}$ and $\twoheaduparrow G = \{x \in L \mid G \ll x\}$. For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P for which $\sup D$ exists, $y \leq \sup D$ implies $D \cap \uparrow x \neq \emptyset$. The poset P is *continuous* if for all $x \in P$, $\twoheaddownarrow x = \{y \in L \mid y \ll x\}$ is directed and $x = \sup \twoheaddownarrow x$.

A poset P is *directed complete* if $\sup D$ exists for all $D \subseteq^{\uparrow} P$. A directed complete poset will be called a dcpo.

A subset A of a topological space is saturated if A is the intersection of all open sets containing A. For a topological space X, the set of all compact saturated subsets of X is denoted by $Q(X)$. We write $\mathfrak{K} \subseteq_{flt} Q(X)$ represents that $\mathfrak{K}$ is filtered. We denote the set of all open sets of space X by $\mathcal{O}(X)$. On $Q(X)$, we consider the *upper Vietoris topology* generated by the sets $\Box U = \{K \in Q(X) \mid K \subseteq U\}$, where $U \in \mathcal{O}(X)$.

Definition 2.1 ([6]) A poset P is said to have *one-step closure* if $cl(A) = A'$ holds for any $A \subseteq P$, where $A' = \{x \in P \mid \exists D \subseteq^{\uparrow} \downarrow A, x = \sup D\}$.

Definition 2.2 ([2]) A poset P is meet continuous if for any $x \in P$ and any directed set D of P with $\sup D$ existing, $x \leq \sup D$ implies $x \in cl(\downarrow D \cap \downarrow x)$.

Remark 2.3 For a semilattice L, one can prove that it is *meet continuous* if and only if it satisfies $\inf\{x, \sup D\} = \sup_{d \in D} \inf\{x, d\}$ for any $x \in L$ and any directed set $D \subseteq L$ with $\sup D$ existing.

Definition 2.4 ([2]) A poset P is *quasicontinuous*, if for every $x \in P$,

(1) $fin(x) = \{F \mid F \in Fin(P), F \ll x\}$ is a directed family;

(2) $\uparrow x = \bigcap_{F \in fin(x)} \uparrow F$ for any $x \in P$.

A quasicontinuous dcpo is called a quasicontinuous domain.

For any quasicontinuous domain P, the family $\{\twoheaduparrow F : F \subseteq P \text{ is finite}\}$ is a base of the Scott topology on P ([2]).

Definition 2.5 ([2]) A space X is *well-filtered* if for each filter basis $\mathcal{C}$ of compact saturated sets of X and each open set U with $\bigcap \mathcal{C} \subseteq U$, there is a $K \in \mathcal{C}$ such that $K \subseteq U$.

Definition 2.6 ([1]) The set $\mathbb{R}$ of all real numbers equipped with the topology having $\{[x, y) \mid x < y, x, y \in \mathbb{R}\}$ as a base is called the *Sorgenfrey line*, which is denoted by $\mathbb{R}_l$.

3 Weak one-step closure

By [6], every continuous poset has one-step closure. However, a quasicontinuous poset may not have one-step closure. In this section, we consider a weaker property, called weaker one-step closure. We prove that every quasicontinuous domain has the weak one-step closure, but a quasicontinuous poset need not have this property.

Definition 3.1 A poset P is said to have the *weak one-step closure* if for any $A \subseteq P$, it holds that $cl(A) = A''$, where $A'' = \{x \in P \mid \exists D \subseteq^{\uparrow} \downarrow A,\ x \leq \sup D\}$

Remark 3.2 In [7], Zhao introduced the Definition 3.1 for an arbitrary set system, and called it one-step closure. To be consistent with the paper [6], here we call this property weak one-step closure.

Theorem 3.3 *Every quasicontinuous dcpo has weak one-step closure.*

Proof. *It suffices to show that $cl(A) \subseteq A''$ for any subset A of L. To this end, let $x \in cl(A)$, $F \in fin(L)$ with $x \in \Uparrow F$. Then $\Uparrow F$ is Scott open as L is quasicontinuous. Hence $\Uparrow F \cap A \neq \emptyset$, which implies that $F \cap \downarrow A \neq \emptyset$. Thus $(F \cap \downarrow A)_{F \in fin(x)}$ is a filtered family (with respective to the Smyth preorder) of nonempty finite subsets of L. By Rudin's Lemma ([2]), there exists a directed subset D of $\bigcup_{F \in fin(x)} F \cap \downarrow A$ such that $D \cap (F \cap \downarrow A) \neq \emptyset$ for any $F \in fin(x)$. Also, since L is a quasicontinuous domain, $\{\Uparrow F \mid F \in fin(x)\}$ is a neighborhood basis of x. This indicates that $x \in cl(D) = \downarrow \sup D$. Note that $D \subseteq \downarrow A$. We conclude that $x \in A''$. Hence $cl(A) \subseteq A''$.* □

The following example shows that the converse conclusion of Theorem 3.3 is not true.

Example 3.4 Let $L = (\mathbb{N} \times \mathbb{N}) \cup \{\top\}$. Define order $\leq$ on L as follows:

(i) $(m, n) \leq (s, t)$ if and only if $m = s$ and $n \leq t$;

(ii) $x \leq \top$ for all $x \in L$.

It is well known that L is a dcpo and not quasicontinuous. However, we can easily verify that L has weak one-step closure.

Note that this dcpo L does not have one-step closure.

The dcpo L is illustrated in Figure 1.

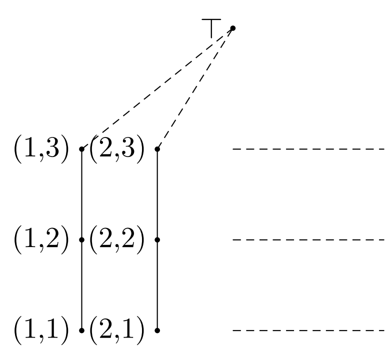

Fig.1. A non-quasicontinuous domain that has weak one-step closure.

The following example shows that a quasicontinuous poset may not have weak one-step closure.

Example 3.5 Let $L = (\mathbb{N} \times (\mathbb{N} \cup \{\omega\})) \cup \mathbb{N}$. We define an order $\leq$ on L as follows:

For any $x, y \in L$, $x \leq y$ if and only if one of the following holds:

(i) $x = (m, n_1), y = (m, n_2), n_1 \leq n_2$;
(ii) $x = (m, n_1), y = (m, \omega)$;
(iii) $x, y \in \mathbb{N}$ and $x \leq y$ in $\mathbb{N}$;
(iv) $x = (m, n), y \in \mathbb{N}, y \geq n, m \geq 2$;
(v) $x = (1, n_1), y = (m_2, \omega), m_2 \geq n_1$;
(vi) $x = (m_1, n), y = (m_2, n), m_1 \leq m_2, m_1 \geq 2$.

L can be illustrated in Figure 2. Then L is a quasicontinuous poset, but L does not have weak one-step closure.

To see this, first note that $(1, \omega) \in cl(\mathbb{N}) = L$ and $(1, \omega) \notin \mathbb{N}''$. Hence, L does not have weak one-step closure. It remains to show that L is quasicontinuous.

(i) For $(1, \omega)$, we have $\{(1, n) \mid n \in \mathbb{N}\} \subseteq^{\uparrow} \Downarrow(1, \omega)$ and $(1, \omega) = \sup_{n\in\mathbb{N}}(1, n)$.
(ii) For each $(1, n) \in L$. Let $F_{n,m} = \{\{(1, n), (2, m)\} \mid m \in \mathbb{N}\}$. Then $\{F_{n,m} \mid m \in \mathbb{N}\} \subseteq fin(1, n)$ and is a filtered base with $\uparrow(1, n) = \bigcap_{n\in\mathbb{N}} \uparrow F_{n,m}$.
(iii) For each $(m, n) \in L$ with $m \in \mathbb{N}$ and $m \geq 2$, we see easily that $(m, n) \ll (m, n)$. In addition, each (m, ω) with $m \geq 2$ is the supremum of the directed set $\{(m, n) : n \in \mathbb{N}\}$ of compact elements.

All these together show that L is quasicontinuous.

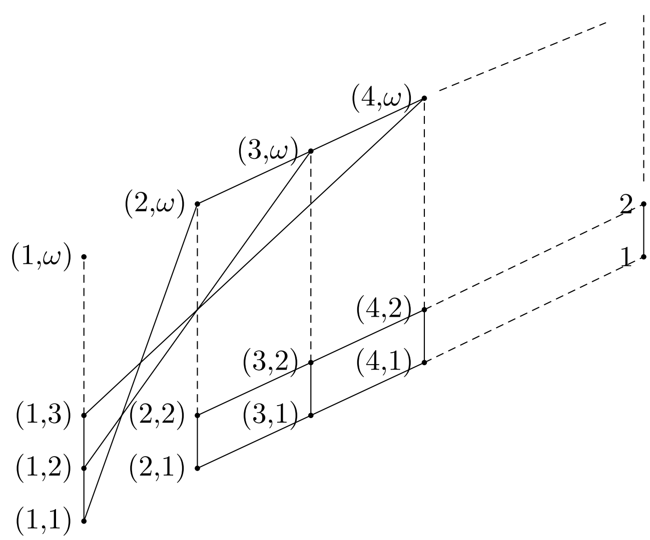

Fig.2. A quasicontinuous poset does not have weak one-step closure.

4 One-step closure

In [6], Zou, Li and Ho showed that every continuous poset has one-step closure. They asked whether L is continuous if it has one-step closure. We now give a counterexample for their problem. We begin with a lemma which is crucial for further study.

Lemma 4.1 *If X is a well-filtered space and $Q(X)$ endowed with the upper Vietoris topology is first-countable, then $(Q(X), \supseteq)$ has one-step closure.*

Proof. *Let $\mathcal{A} \subseteq Q(X)$ and $K \in cl(\mathcal{A})$. The fact that $Q(X)$ equipped with the upper Vietoris topology is first-countable implies that there exists a countable neighborhood basis $\mathcal{B}_K = \{\Box U_n \mid n \in \mathbb{N}\}$ of K and $\Box U_{n+1} \subseteq \Box U_n$ for any $n \in \mathbb{N}$.*

Claim 1: $\Box U \subseteq \Box V$ implies $U \subseteq V$ for any $U, V \in \mathcal{O}(X)$.

Let $x \in U$. Then $\uparrow x \in \Box U \subseteq \Box V$. In other words $\uparrow x \subseteq V$. So $U \subseteq V$ holds.

From Theorem 5.8 in [5], we know that the upper Vietoris topology coincides with the Scott topology on $(Q(X), \supseteq)$. It follows that $\Box U_n \cap \mathcal{A} \neq \emptyset$ for any $\Box U_n \in \mathcal{B}_K$ due to the assumption that $K \in cl(\mathcal{A})$. Choose $K_n \in \Box U_n \cap \mathcal{A}$ for any $n \in \mathbb{N}$. We define $Q_n = K \cup \bigcup_{m \geq n} K_m$ for any $n \in \mathbb{N}$.

Claim 2: $Q_n \in Q(X)$ for each $n \in \mathbb{N}$.

As a union of saturated sets, Q_n is a saturated. It suffices to verify that Q_n is compact. Let $\{W_i : i \in I\}$ be a family of open sets of X such that $Q_n \subseteq \bigcup_{i \in I} W_i$. Then $K \subseteq \bigcup_{i \in I} W_i$. As K is compact, there exists $F_1 \in Fin(I)$ such that $K \subseteq \bigcup_{i \in F_1} W_i$. Then, there exists $\Box U_{n_0} \in \mathcal{B}_K$ such that $K \in \Box U_{n_0} \subseteq \Box \bigcup_{i \in F_1} W_i$. We consider the following two cases:

Case 1. $n_0 \leq n$: For any $m \geq n \geq n_0$, then $K_m \subseteq U_m \subseteq U_{n_0}$ by Claim 1. Hence, $Q_n \subseteq U_{n_0} \subseteq \bigcup_{i \in F_1} W_i$.

Case 2. $n_0 > n$: We can obtain that $\bigcup_{m \geq n_0} K_m \subseteq U_{n_0} \subseteq \bigcup_{i \in F_1} W_i$ by the similar proof to Case 1. Note that $\bigcup_{i=n}^{n_0-1} K_i \in Q(X)$ and $\bigcup_{i=n}^{n_0-1} K_i \subseteq \bigcup_{i \in I} W_i$. This means that there exists $F_2 \in Fin(I)$ such that $\bigcup_{i=n}^{n_0-1} K_i \subseteq \bigcup_{i \in F_2} W_i$. Therefore, $Q_n \subseteq \bigcup_{i \in F_1 \cup F_2} W_i$.

Claim 3: $K = \sup_{n \in \mathbb{N}} Q_n = \bigcap_{n \in \mathbb{N}} Q_n$.

It is easy to see that $K \subseteq \bigcap_{n \in \mathbb{N}} Q_n$. For the converse, suppose $x \in \bigcap_{n \in \mathbb{N}} Q_n$. We claim that $x \in K$. Assume $x \notin K$. This manifests $\downarrow x \cap K = \emptyset$. In other words, $K \subseteq X \backslash \downarrow x$. It follows that there exists $n \in \mathbb{N}$ such that $K \in \Box U_n \subseteq \Box X \backslash \downarrow x$. Through Claim 2, we can conclude that $x \in Q_n \subseteq U_n \subseteq X \backslash \downarrow x$, which contradicts $x \in \downarrow x$.

Note that $K_n \in \mathcal{A}$ for any $n \in \mathbb{N}$ and $Q_n \leq K_n$ (with respect to the reverse inclusion order). So $(Q_n)_{n \in \mathbb{N}}$ is a directed subset of $\downarrow \mathcal{A}$ whose supremum equals K. Hence, $Q(X)$ has one-step closure. □

The conclusion given in the next theorem answers the question from Zou et al..

Theorem 4.2 *For the Sorgenfrey line $\mathbb{R}_l$, $Q(\mathbb{R}_l)$ has one-step closure and $Q(\mathbb{R}_l)$ is not continuous.*

Proof. *By Example 5.18 of [5], we know that the poset $Q(\mathbb{R}_l)$ is not continuous. The space $\mathbb{R}_l$ is Hausdorff, thus well-filtered (every Hausdorff space is sober and every sober space is well-filtered). Hence, by Lemma 4.1, $Q(\mathbb{R}_l)$ has one-step closure.* □

5 The relationship between weak one-step closure and one-step closure

In this section, we investigate the relationship between weak one-step closure and one-step closure.

The following lemma and example justify the term "weak one-step closure".

Lemma 5.1 *If a poset P has one-step closure, then it has weak one-step closure.*

Proof. *It suffices to prove that $cl(A) = A''$ for any subset A of P. From the definition of one-step closure, we have $cl(A) = A'$. One sees obviously that $A'' \subseteq cl(A)$. Let $x \in cl(A)$. Then $x \in A'$. It follows that there exists $D \subseteq^{\uparrow} \downarrow A$ such that $x = \sup D$, i.e., $x \in A''$.* □

The converse of Lemma 5.1 is not true.

Example 5.2 Let $L = \mathbb{N} \cup \{\omega, a\}$, where $\mathbb{N}$ denotes all natural numbers. We define an order $\leq$ on L by $x \leq y$ if and only if:

(i) $x, y \in \mathbb{N}$ and $x \leq y$ holds in $\mathbb{N}$, or

(ii) $x \in L$ and $y = \omega$.

Then L can be easily illustrated in Figure 3. and $\mathbb{N}' = \mathbb{N} \cup \{\omega\} \subsetneqq \mathbb{N}'' = L$. Thus L does not have one-step closure. But it is easy to check that L has weak one-step closure.

It is then natural to wonder under what conditions, a poset has one-step closure if it has weak one-step closure. We shall prove that if a poset has weak one-step closure, then it has one-step closure if and only if it is meet continuous.

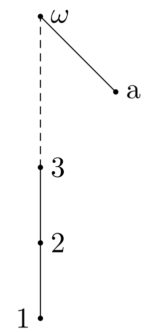

Fig. 3. $\mathbb{N}'' \neq \mathbb{N}'$.

Lemma 5.3 *Let L be a poset. Then the following statements are equivalent:*

(1) A' is a lower set for any $A \subseteq L$;

(2) D' is a lower set for any directed subset D of L.

Proof. (1) $\Rightarrow$ (2) *is trivial.*

(2) $\Rightarrow$ (1) *Assume $x \leq y \in A'$. Then there exists $D \subseteq^{\uparrow} \downarrow A$ such that $y = \sup D$. This means that $x \leq y \in D'$. It follows that $x \in \downarrow D' = D'$. So we have that there exists a directed subset E of $\downarrow D$ such that $x = \sup E$. Note that $E \subseteq \downarrow D \subseteq \downarrow A$. Therefore, $x \in A'$.* □

If L has one-step closure, then for any subset $A \subseteq L$, $cl(A) = A'$, so it is a lower set.

In [6], Zou, Li and Ho proved that L is meet continuous if L has one-step closure. We now deduce this result using a weak assumption.

Lemma 5.4 *Let L be a poset. If $D' = \downarrow D'$ for any $D \subseteq^{\uparrow} L$, then L is meet-continuous.*

Proof. *Let $x \in L$, $D \subseteq^{\uparrow} L$ with $\sup D$ existing. If $x \leq \sup D$, then $x \in \downarrow D' = D'$. This means that there exists a directed subset E of $\downarrow D$ such that $x = \sup E$. Note that $E \subseteq \downarrow x \cap \downarrow D$. This implies that $x \in cl(\downarrow x \cap \downarrow D)$. Therefore, L is meet-continuous.* □

Corollary 5.5 *Every poset with one-step closure is meet continuous.*

Corollary 5.6 *Let L be a meet continuous semilattice. Then D' is a lower set for any directed subset D of L. Moreover, if L has weak one-step closure, then L has one step closure.*

Proof. *From Lemma 5.8, it suffices to prove that D' is a lower set for any directed subset D of L. Suppose $x \leq y \in D'$. Then there exists a directed subset E of $\downarrow D$ with $y = \sup E$. The fact that L is a meet continuous semilattice implies that $x = \sup_{e \in E} \inf\{x, e\}$. It is noteworthy that $\{\inf\{x, e\} \mid e \in E\}$ is a directed subset of $\downarrow D$. This means that $x \in D'$. Therefore, D' is a lower set.* □

Corollary 5.7 *Let L be a meet continuous sup-semilattice. Then D' is a lower set for any directed subset D of L.*

Proof. *Let $x \leq y \in D'$. This means that there exists a directed subset E of $\downarrow D$ such that $y = \sup E$. Since L is meet continuous, $x \in cl(\downarrow x \cap \downarrow E)$. It follows that $\downarrow x = cl(\downarrow x \cap \downarrow E)$, and hence $x = \sup(\downarrow x \cap \downarrow E)$. Let $G = \{\sup F : F \subseteq (\downarrow x \cap \downarrow E)$ and F is finite$\}$. Then G is a directed subset of $\downarrow x \cap \downarrow E$ with $\sup G = \sup(\downarrow x \cap \downarrow E)) = x$. Clearly $G \subseteq \downarrow D$, thus $x \in D'$, showing that D' is lower.* □

Lemma 5.8 *Let L be a poset with weak one-step closure. If D' is a lower set for any directed subset D of L, then L has one-step closure.*

Proof. *This follows immediately from Definition 3.1, Definition 2.1 and Lemma 5.3.* □

Lemma 5.9 *Let L be a meet continuous poset with weak one-step closure. Then L has one-step closure.*

Proof. *By Lemma 5.8, it suffices to show that $D^{'}$ is a lower set for any directed $D \subseteq L$. Suppose $x \leq y \in D^{'}$. Then, there exists a directed subset E of $\downarrow D$ such that $y = \sup E$. Since L is meet continuous, we have $x \in cl(\downarrow x \cap \downarrow E)$. Since L has weak one-step closure, there is a directed $K \subseteq \downarrow x \cap \downarrow E$ such that $x \leq \sup K$. But, trivially $\sup K \leq x$, hence $x = \sup K$. In addition, $K \subseteq \downarrow E \subseteq \downarrow D$, so $x \in D^{'}$. Therefore $D^{'}$ is a lower set.* □

From Lemma 5.1 and Lemma 5.9 we deduce the following result.

Theorem 5.10 *A poset has one-step closure if and only if it is meet continuous and has weak one-step closure.*

Since every poset having one-step closure is meet continuous, we have the following natural problem.

Problem 5.11 *Is there a meet continuous poset that does not have one-step closure.*

We have already given in Section 4 an example of a non-continuous poset that has one-step closure. We now confirm that an exact poset with one-step closure is continuous.

Definition 5.12 ([4]) Let x, y be elements of a poset P. We say that x is *weakly way-below* y, denoted by $x \ll_w y$, if for any directed subset D of P for which $\sup D$ exists, $y = \sup D$ implies $D \cap \uparrow x \neq \emptyset$. A poset P is called *exact* if for any $x \in P$, $\Downarrow_w x = \{y \in P \mid y \ll_w x\}$ is directed and $\sup \Downarrow_w x = x$.

Theorem 5.13 *Let L be a poset. Then the following statements are equivalent:*

(1) *L is continuous;*
(2) *L has one-step closure and is exact;*
(3) *$A^{'}$ is a lower set for any $A \subseteq L$ and L is an exact poset.*
(4) *$D^{'}$ is a lower set for any directed subset D of L and L is an exact poset.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) *are all obvious.*

(4) ⇒ (1) *From the definition of exact posets and continuous posets, it suffices to prove that $x \ll_w y$ implies that $x \ll y$ for any $x, y \in L$. Let D be a directed subset of L with $y \leq \sup D$, it follows that $\sup D \in D^{'}$. We have that $y \in D^{'}$ since $D^{'}$ is a lower set. This means that there exists $E \subseteq^{\uparrow} \downarrow D$ such that $y = \sup E$. The assumption that $x \ll_w y$ reveals that $\uparrow x \cap E \neq \emptyset$. Hence, $\uparrow x \cap D \neq \emptyset$.* □

The following is a problem concerning the connections among the concepts of meet continuity, exactness and continuity of posets.

Problem 5.14 *Let L be a meet continuous and exact poset. Must L be continuous?*

Although we cannot solve the above problems, we have the following corollary by Corollary 5.6 and Corollary 5.7.

Corollary 5.15 *Let L be a meet continuous semilattice or sup-semilattice. Then L is continuous iff L is an exact poset.*

Proposition 5.16 *Let L, M be two posets. If L is a Scott retract of M, which has one-step closure, then L has one-step closure.*

Proof. *Since L is a Scott retract of M, we have that there exists two Scott continuous maps $s : L \to M$ and $r : M \to L$ such that $id_L = r \circ s$. Now let $x \in L$, $A \subseteq L$ with $x \in cl(A)$. It follows that $s(x) \in cl(s(A))$ by the Scott continuity of s. We know that there exists $D \subseteq^{\uparrow} \downarrow s(A)$ such that $s(x) = \sup D$ since M has one-step closure. This implies that $x = r \circ s(x) = r(\sup D) = \sup r(D)$ from the Scott continuity of r. Note that $r(D) \subseteq \downarrow r(s(A)) = \downarrow A$. This means that L has one-step closure.* □

Lemma 5.17 *Let L, M be two posets. If L is a Scott retract of M, which has weak one-step closure, then L has weak one-step closure.*

Proof. *The proof is similar to Proposition 5.16.* □

References

[1] R. Engelking, “General Topology”, Polish Scientific Publishers, Warzawa, 1989, ISBN 3885380064.

[2] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, “Continuous Lattices and Domains”, Volume **93** of Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2003. https://doi.org/10.1017/CBO9780511542725

[3] J. Goubault-Larrecq, “Non-Hausdorff Topology and Domain Theory”, Volume **22** of New Mathematical Monographs, Cambridge University Press, 2013. https://doi.org/10.1017/CBO9781139524438

[4] C. Shen, G. Wu, D. Zhao, *Weak Domain Models of T_1 spaces*, Electronic Notes in Theoretical Computer Science, **345** (2019): 219-232. https://doi.org/10.1016/j.entcs.2019.07.025

[5] X. Xu, Z. Yang, *Coincidence of the upper Vietoris topology and the Scott topology*, Topology and its Applications, **288** (2021): 107480. https://doi.org/10.1016/j.topol.2020.107480

[6] Z. Zou, Q. Li, W. Ho, *Domains via approximation operators*, Logical Methods in Computer Science, **14** (2018): 1-17. https://doi.org/10.23638/LMCS-14(2:6)2018

[7] D. Zhao, *Generalizations of Frames and Continuous Lattices*, Ph.D. Dissertation, The University of Cambridge, 1992.