Continuous R-valuations

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Abstract

We introduce continuous R-valuations on directed-complete posets (dcpos, for short), as a generalization of continuous valuations in domain theory, by extending values of continuous valuations from reals to so-called Abelian d-rags R.

Like the valuation monad $\mathbf{V}$ introduced by Jones and Plotkin, we show that the construction of continuous R-valuations extends to a strong monad $\mathbf{V}^R$ on the category of dcpos and Scott-continuous maps. Additionally, and as in recent work by the two authors and C. Théron, and by the second author, B. Lindenhovius, M. Mislove and V. Zamdzhiev, we show that we can extract a commutative monad $\mathbf{V}_m^R$ out of it, whose elements we call minimal R-valuations.

We also show that continuous R-valuations have close connections to measures when R is taken to be $\mathbb{IR}_+^\star$, the interval domain of the extended nonnegative reals: (1) On every coherent topological space, every non-zero, bounded τ-smooth measure μ (defined on the Borel σ-algebra), canonically determines a continuous $\mathbb{IR}_+^\star$-valuation; and (2) such a continuous $\mathbb{IR}_+^\star$-valuation is the most precise (in a certain sense) continuous $\mathbb{IR}_+^\star$-valuation that approximates μ, when the support of μ is a compact Hausdorff subspace of a second-countable stably compact topological space. This in particular applies to Lebesgue measure on the unit interval. As a result, the Lebesgue measure can be identified as a continuous $\mathbb{IR}_+^\star$-valuation. Additionally, we show that the latter is minimal.

Keywords: R-valuations; measures; dcpos; commutative monads.

1   Introduction

The probability of an event is most often than not understood as a real number between 0 and 1, and measures, as well as continuous valuations, take their values in $\overline{\mathbb{R}}_+$, the set of non-negative real numbers extended with $+\infty$. What is there that is so special with real numbers, and can we replace $\overline{\mathbb{R}}_+$ by some elements in some other structure? The question was once asked by Vincent Danos to the first author, and came back to the authors in an attempt to formulate an alternative to measures and continuous valuations with values taken as *exact reals*, in the sense of Real PCF [10,9,8,24] for example. Exact reals are modeled there as *intervals* that enclose the true value that is intended, and computation proceeds by refining these intervals further and further. Indeed, one of the points of this paper is that we can extend continuous valuations to an interval-valued form of continuous valuations, with an interval-valued integration theory. In addition, this also leads us to commutative valuations monads with intervals as values on the category of dcpos and Scott-continuous maps.

We should warn the reader that such an endeavor is probably useless for computation purposes. In the setting of type 2 theory of effectivity, Weihrauch has shown that, under reasonable assumptions, the map

that sends a representable measure μ on $[0,1]$ to $\mu[0,1/2)$ cannot be continuous if the target space $[0,1]$ is given the Scott topology of the reverse ordering $\geq$ [27, Theorem 2.7]. (It is continuous with respect to the usual ordering $\leq$. In passing, type 2 theory of effectivity on the reals does not differ much from ordinary domain-based notions of computability, as Schulz has shown [25].) This roughly means that if we insist on representing $\mu[0,1/2)$ as *precise* intervals, namely as intervals $[a,a]$ with the same left and right bounds, then the rightmost a will evolve in a discontinuous manner.

Despite this, we will show in Section 7 that quite a lot of measures (in the ordinary sense) have representations as interval-valued "measures" with *precise* intervals; see Remark 7.8, in particular. Before that, we will have to define what we mean by "measure" with values in a domain of intervals. We will start (after recapitulating some preliminary notions and results in Section 2) by giving a pretty general possible answer to V. Danos' question in Section 3: we may safely replace $\overline{\mathbb{R}}_+$ by any structure of a kind that we call an *Abelian d-rag*, which is a weaker form of Abelian semiring, with a compatible ordering that turns it into a dcpo. This will allow us to define a notion of *continuous R-valuation* on a space X, for any Abelian d-rag R, in Section 4. The obvious definition would be as a function from the open subsets of X satisfying certain requirements, but those requirements have proved elusive, especially in Abelian d-rags where the additive zero 0 differs from the bottom element $\perp$, as with the domain of intervals. For example, would you define the measure of the empty set as 0 or as $\perp$? One is needed for algebraic reasoning, so that adding the measure of the empty set does nothing; the other is needed for approximation purposes, e.g., in order to define the integral of a function f as the supremum of simpler sums. We sidestep the issue by defining our continuous R-valuations as being directly the functionals that one would usually obtain by defining an integral out of a measure. In Section 5, we show that continuous R-valuations, much like continuous valuations [18,17], can be organized to form a strong monad on the category **Dcpo** of dcpos and Scott-continuous maps. Furthermore, as in [14] and [16], one can carve out a *commutative* monad of so-called *minimal R*-valuations from the latter. We start to examine the relationship between measures and continuous R-valuations when R is either $\overline{\mathbb{R}}_+$ or the interval domain $\mathbb{IR}_+^\star$ in Section 6. That section is devoted to a few simple facts, and notably to the fact that every continuous $\mathbb{IR}_+^\star$-valuation induces an ordinary continuous $(\overline{\mathbb{R}}_+$-$)$valuation, which we call it view from the left. In Section 7, we will see that every non-zero, bounded τ-smooth measure μ on a coherent topological space gives rise to a continuous $\mathbb{IR}_+^\star$-valuation $\widetilde{\mu}$ in a natural way, and that $\widetilde{\mu}$ is *precise* in the sense alluded to above. In Section 8, under slightly different assumptions, we study the continuous $\mathbb{IR}_+^\star$-valuations that approximate a given, not necessarily bounded, measure, and we show that there is a *most precise* one; it so happens that this is $\widetilde{\mu}$, once again. In all those cases, there is no reason why $\widetilde{\mu}$ should be minimal. In Section 9, we illustrate the question with the Lebesgue measure λ on $[0,1]$ and its associated continuous $\mathbb{IR}_+^\star$-valuation $\widetilde{\lambda}$. We note that $\widetilde{\lambda}$ is not minimal. However, we will show that replacing λ by its image measure under the inclusion of $[0,1]$ into a dcpo of intervals $\mathbb{IR}^\star$ *does* yield a minimal $\mathbb{IR}_+^\star$-valuation. We conclude in Section 10.

2 Preliminaries

We refer to [3] for basics of measure theory, and to [1,11,12] for basics of domain theory and topology.

Measure theory.

A σ-*algebra* on a set X is a collection of subsets closed under countable unions and complements. A *measurable space* X is a set with a σ-algebra Σ_X. The elements of Σ_X are usually called the *measurable subsets* of X.

A *measure* μ on X is a σ-additive map from Σ_X to $\overline{\mathbb{R}}_+ \overset{\text{def}}{=} \mathbb{R}_+ \cup \{+\infty\}$, where $\overline{\mathbb{R}}_+$ is the set of extended non-negative real numbers . The property of σ-*additivity* means that, for every countable family of pairwise disjoint sets E_n, $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$. (Here n ranges over any subset of $\mathbb{N}$, possibly empty.)

A *measurable map* $f\colon X \to Y$ between measurable spaces is a map such that $f^{-1}(E) \in \Sigma_X$ for every $E \in \Sigma_Y$. The *image measure* $f[\mu]$ of a measure μ on X is defined by $f[\mu](E) \overset{\text{def}}{=} \mu(f^{-1}(E))$.

The σ-algebra $\Sigma(A)$ *generated by* a family A of subsets of X is the the smallest σ-algebra containing A. The *Borel σ-algebra* on a topological space is the σ-algebra generated by its topology. The *standard topology* on $\overline{\mathbb{R}}_+$ is generated by the intervals $[0,b[$, $]a,b[$ and $]a,+\infty$, with $0 < a < b < +\infty$. Its Borel

σ-algebra is also generated by the intervals $]a, +\infty]$ along (the Scott-open subsets, see below). Hence a measurable map $h \colon X \to \overline{\mathbb{R}}_+$ is a map such that $h^{-1}(]t, +\infty]) \in \Sigma$ for every $t \in \mathbb{R}$. Its *Lebesgue integral* can be defined elegantly through *Choquet's formula*: $\int_X h d\mu \stackrel{\text{def}}{=} \int_0^{+\infty} \mu(h^{-1}(]t, +\infty])) dt$, where the right-hand integral is an ordinary Riemann integral.

This formula makes the following *change-of-variables formula* an easy observation: for every measurable map $f \colon X \to Y$, for every measurable map $h \colon Y \to \overline{\mathbb{R}}_+$, $\int_Y h df[\mu] = \int_X (h \circ f) d\mu$.

The *monotone convergence theorem* states that, given any measure μ on a measurable space X, given any sequence $(h_n)_{n \in \mathbb{N}}$ of measurable maps from X to $\overline{\mathbb{R}}_+$ that is pointwise monotonic, their pointwise supremum h is measurable, and $\int_X h d\mu = \sup_{n \in \mathbb{N}} \int_X h_n d\mu$. If $(h_n)_{n \in \mathbb{N}}$ is antitonic instead, then a similar theorem holds provided that $\int_X h_n d\mu < +\infty$ for some $n \in \mathbb{N}$ (but not in general): the pointwise infimum h is measurable, and $\int_X h d\mu = \inf_{n \in \mathbb{N}} \int_X h_n d\mu$.

A practical way of building measures is *Carathéodory's measure existence theorem*, which is the following. A *semi-ring* $\mathcal{R}$ on X is a collection of subsets of X that is closed under finite intersections, and such that the complement of every element of $\mathcal{R}$ can be written as a finite disjoint union of elements of $\mathcal{R}$. A map $\mu \colon \mathcal{R} \to \overline{\mathbb{R}}_+$ is called σ-*additive*, extending the definition given above, if and only if for every countable (possibly empty) collection of pairwise distinct elements E_n of $\mathcal{R}$ whose union E is also in $\mathcal{R}$, $\mu(E) = \sum_n \mu(E_n)$. Then μ extends to a measure on some σ-algebra containing $\mathcal{R}$. A first use of this theorem is to establish the existence of *Lebesgue measure* λ on $\mathbb{R}$, defined so that $\lambda(]a, b[) = b - a$ for every open bounded interval $]a, b[$.

A measure μ on X is *bounded* if and only if $\mu(X) < +\infty$. A measure μ is σ-*finite* if there is a sequence $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n \subseteq \cdots$ of measurable subsets of X whose union is X and such that $\mu(E_n) < +\infty$ for every $n \in \mathbb{N}$. A π-*system* Π on a set X is a family of sets closed under finite intersections. If X is a measurable space such that $\Sigma_X = \Sigma(\Pi)$, any two σ-finite measures that agree on Π also agree on Σ_X. In particular, Lebesgue measure on $\mathbb{R}$ is uniquely defined by the specification $\lambda(]a, b[) = b - a$.

Domain theory and topology.

A *dcpo* is a poset in which every directed family D has a supremum $\sup D$. A prime example is $\mathbb{IR}^\star$, the poset of closed intervals $[a, b]$ with $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $a \leq b$, ordered by reverse inclusion. Every directed family $([a_i, b_i])_{i \in I}$ has a supremum $\bigcap_{i \in I} [a_i, b_i] = [\sup_{i \in I} a_i, \inf_{i \in I} b_i]$. Among them, we find the *total* numbers $a \in \mathbb{R} \cup \{-\infty, +\infty\}$, which are equated with the maximal elements $[a, a]$ in $\mathbb{IR}^\star$.

Another example is $\overline{\mathbb{R}}_+$, with the usual ordering. We will also consider $\mathbb{IR}_+^\star$, the subdcpo of $\mathbb{IR}^\star$ consisting of its elements of the form $[a, b]$ with $a \geq 0$.

We will also write $\leq$ for the ordering on any poset. In the example of $\mathbb{IR}^\star$, $\leq$ is $\supseteq$. The *upward closure* $\uparrow A$ of a subset A of a poset X is $\{y \in X \mid \exists x \in A, x \leq y\}$. The *downward closure* $\downarrow A$ is defined similarly. A set A is *upwards closed* if and only if $A = \uparrow A$, and *downwards closed* if and only if $A = \downarrow A$. A subset U of a dcpo X is *Scott-open* if and only if it is upwards closed and, for every directed family D such that $\sup D \in U$, some element of D is in U already. The Scott-open subsets of a dcpo X form its *Scott topology*.

The *way-below* relation $\ll$ on a poset X is defined by $x \ll y$ if and only if, for every directed family D with a supremum z, if $y \leq z$, then x is less than or equal to some element of D already. We write $\Uparrow x$ for $\{y \in X \mid x \ll y\}$, and $\downdownarrows y$ for $\{x \in X \mid x \ll y\}$. A poset X is *continuous* if and only if $\downdownarrows x$ is directed and has x as supremum for every $x \in X$. A *basis* B of a poset X is a subset of X such that $\downdownarrows x \cap B$ is directed and has x as supremum for every $x \in X$. A poset X is continuous if and only if it has a basis (namely, X itself). A poset is ω-*continuous* if and only if it has a countable basis. Examples include $\overline{\mathbb{R}}_+$, with any countable dense subset (with respect to its standard topology), such as the rational numbers in $\overline{\mathbb{R}}_+$, or the *dyadic numbers* $k/2^n$ ($k, n \in \mathbb{N}$); or $\mathbb{IR}^\star$ and $\mathbb{IR}_+^\star$, with the basis of intervals $[a, b]$ where a and b are both dyadic or rational.

We write $\mathcal{O}X$ for the lattice of open subsets of a topological space X. This applies to dcpos X as well, which will always be considered with their Scott topology. The continuous maps $f \colon X \to Y$ between two dcpos coincide with the *Scott-continuous* maps, namely the monotonic (order-preserving) maps that preserve all directed suprema. We write $\mathcal{L}X$ for the space of continuous maps from a topological space X to $\overline{\mathbb{R}}_+$, the latter with its Scott topology, as usual. Such maps are usually called *lower semicontinuous*, or *lsc*, in the mathematical literature. Note that $\mathcal{L}X$, with the pointwise ordering, is a dcpo.

3

There are several ways in which one can model probabilistic choice. The most classical one is through measures. A popular alternative used in domain theory is given by *continuous valuations* [18,17]. A continuous valuation is a Scott-continuous map $\nu\colon \mathcal{O}X \to \overline{\mathbb{R}}_+$ such that $\nu(\emptyset) = 0$ (*strictness*) and, for all $U, V \in \mathcal{O}X$, $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ (*modularity*). There is a notion of integral $\int_{x \in X} h(x) d\nu$, or briefly $\int h d\nu$, for every $h \in \mathcal{L}X$, which can again be defined by a Choquet formula. The map $h \in \mathcal{L}X \mapsto \int h d\nu$ is Scott-continuous and *linear*. By definition, a linear map $G\colon \mathcal{L}X \to \overline{\mathbb{R}}_+$ satisfies $G(h + h') = G(h) + G(h')$ and $G(\alpha.h) = \alpha.G(h)$ for all $\alpha \in \mathbb{R}_+$, $h, h' \in \mathcal{L}X$. Conversely, any Scott-continuous linear map $G\colon \mathcal{L}X \to \overline{\mathbb{R}}_+$ is of the form $h \mapsto \int h d\nu$ for a unique continuous valuation ν, given by $\nu(U) \stackrel{\text{def}}{=} G(\chi_U)$, where χ_U is the characteristic map of U ($\chi_U(x) \stackrel{\text{def}}{=} 1$ if $x \in U$, 0 otherwise).

3   Rags, d-rags and continuous d-rags

Definition 3.1 A *rag* is a tuple $(R, 0, +, 1, \times)$ (or simply R) where $(R, 0, +)$ is an Abelian monoid, $(R, 1, \times)$ is a monoid, and $\times$ distributes over $+$. An *Abelian rag* is a rag whose multiplication $\times$ is commutative.

A *semi-ring*, or *rig*, is a rag which satisfies the extra law $0 \times r = r \times 0 = 0$. $\overline{\mathbb{R}}_+$, for example, is an Abelian rig, where $+$ and $\times$ are as usual, modulo the convention that $0 \times (+\infty) = 0$. We will see that $\mathbf{I}\mathbb{R}_+^\star$ is a rag, but not a rig.

We also need some topological structure.

Definition 3.2 A *d-rag* is a rag R together with an ordering that makes it a dcpo, in such a way that $+$ and $\times$ are Scott-continuous.

$\overline{\mathbb{R}}_+$ is a d-rag. In order to turn $\mathbf{I}\mathbb{R}_+^\star$ into a d-rag, we define its 0 element as $[0, 0]$; addition by $[a, b] + [c, d] \stackrel{\text{def}}{=} [a + c, b + d]$; its 1 element as $[1, 1]$; and product by $[a, b] \times [c, d] \stackrel{\text{def}}{=} [a \cdot_\ell c, b \cdot_r d]$. The operations $\cdot_\ell$ and $\cdot_r$ are product operations (for the ℓeft and $r$ight part, respectively), and are defined so that $x \cdot_\ell y$ and $x \cdot_r y$ are equal to the usual product xy unless one of x, y is equal to 0 and the other is equal to $+\infty$. We need two distinct, left and right, product operations in order to ensure Scott-continuity, as we now explain. We must define $0 \cdot_\ell (+\infty)$ ($= (+\infty) \cdot_\ell 0$) as 0, since $0 \cdot_\ell (+\infty)$ must be equal to $\sup_{r \in \mathbb{R}_+} 0 \cdot_\ell r = 0$. Symmetrically, we must define $0 \cdot_r (+\infty)$ ($= (+\infty) \cdot_r 0$) as $+\infty$, because $0 \cdot_r (+\infty)$ must be equal to $\inf_{r > 0} r \cdot_r (+\infty) = +\infty$. With those choices, we have the following easily checked fact.

Lemma 3.3 $\mathbf{I}\mathbb{R}_+^\star$ *is an Abelian d-rag.*

4   Continuous R-Valuations

Let R be a fixed Abelian d-rag. One might be tempted to define continuous R-valuations on a space X as Scott-continuous maps from $\mathcal{O}X$ to R satisfying some appropriate forms of strictness and modularity, but, as we have argued in the introduction, this is fraught with difficulties when the additive unit is not the least element of R.

Since continuous valuations on X correspond bijectively to linear Scott-continuous maps from $\mathcal{L}X$ to $\overline{\mathbb{R}}_+$, another route is to define continuous R-valuations as certain maps from a variant of $\mathcal{L}X$ to R instead of $\overline{\mathbb{R}}_+$. As we will see, this leads to a streamlined theory.

Given any space X, let $\mathcal{L}^R X$ be the dcpo of all continuous maps from X to R, with the pointwise ordering. With pointwise addition and multiplication, $\mathcal{L}^R X$ is also an Abelian d-rag.

Definition 4.1 [Continuous R-valuation] A *continuous R-valuation* on a space X is a Scott-continuous map $\nu\colon \mathcal{L}^R X \to R$ that is *linear* in the sense that $\nu(\mathbf{a} \times h) = \mathbf{a} \times \nu(h)$ (*homogeneity*) and $\nu(h + h') = \nu(h) + \nu(h')$ (*additivity*) for all $\mathbf{a} \in R$, $h, h' \in \mathcal{L}^R X$. We write $\mathbf{V}^R X$ for the dcpo of all continuous R-valuations on X, with the pointwise ordering.

Remark 4.2 When $R = \overline{\mathbb{R}}_+$, $\mathcal{L}^R X = \mathcal{L}X$, so that $\mathbf{V}^R X$ can be equated with the dcpo $\mathbf{V}X$ of ordinary continuous valuations.

In order to help understand the definition, it is profitable to use the integral notation $\int h d\nu$ to mean $\nu(h)$. Hence Definition 4.1 requires that $\int (\mathbf{a} \times h) d\nu = \mathbf{a} \times \int h d\nu$ and $\int (h + h') d\nu = \int h d\nu + \int h' d\nu$.

Beware that the constant 0 map from $\mathcal{L}^R X$ to R is *not* a continuous R-valuation, unless R is a rig: homogeneity would imply $0 = \mathbf{a} \times 0$, which fails in $\mathbf{I}\mathbb{R}_+^\star$, for example. Also, we do not require $\nu(\mathbf{0}) = 0$ in Definition 4.1, where $\mathbf{0}$ is the constant 0 map. This would be a consequence of homogeneity if R were a rig.

Addition and multiplication by scalars in R are defined pointwise on $\mathbf{V}^R X$. This allows us to make sense of the following definition.

Definition 4.3 The *R-Dirac mass at* $x \in X$ is the continuous R-valuation $\delta_x \colon h \in \mathcal{L}^R X \mapsto h(x) \in R$.

An *elementary R-valuation* on X is a continuous R-valuation of the form $\sum_{i=1}^n r_i \times \delta_{x_i}$, where $n \geq 1$, each r_i is in R, and mapping each $h \in \mathcal{L}^R X$ to $\sum_{i=1}^n r_i \times h(x_i)$.

We write $\mathbf{V}_{\mathrm{f}}^R X$ for the poset of elementary R-valuations on X, and $\mathbf{V}_m^R X$ for the inductive closure of $\mathbf{V}_{\mathrm{f}}^R X$ in $\mathbf{V}^R X$. The elements of $\mathbf{V}_m^R X$ are called the *minimal R-valuations* on X.

Our definition of the R-Dirac mass reads $\int h d\delta_x = h(x)$ in integral notation.

Remark 4.4 A *simple valuation* on X is one of the form $\sum_{i=1}^n r_i \delta_{x_i}$, where $n \in \mathbb{N}$, each point x_i is in X, and each coefficient r_i is in $\mathbb{R}_+$. While continuous valuations can be equated with continuous R-valuations with $R = \overline{\mathbb{R}}_+$ (Remark 4.2), simple valuations and elementary R-valuations are closely related but different concepts, even when $R = \overline{\mathbb{R}}_+$. First, $r\delta_x$ is an elementary $\overline{\mathbb{R}}_+$-valuation even when $r = +\infty$, but is a simple valuation only if $r < +\infty$. Second, we require $n \geq 1$ in the definition of elementary R-valuations, but n can be equal to 0 in the definition of a simple valuation. The reason why we require $n \geq 1$ is that the constant 0 map is not a continuous R-valuation in general, as noticed above.

The *inductive closure* of a subset A of a dcpo Z is the smallest subset of Z that contains A and is closed under directed suprema. It is obtained by taking all directed suprema of elements of A, all directed suprema of elements obtained in this fashion, and proceeding this way transfinitely.

A *pointed* dcpo is one with a least element $\bot$.

Proposition 4.5 *Let R be an Abelian d-rag with a least element $\bot$ that is absorbing for multiplication, viz., $\bot \times \mathbf{a} = \bot$ for every $\mathbf{a} \in R$. For every non-empty space X, the constant map $\tilde{\bot} \colon h \in \mathcal{L}^R X \mapsto \bot$ is the least element of $\mathbf{V}^R X$, and also of $\mathbf{V}_m^R X$. Thus, $\mathbf{V}^R X$ and $\mathbf{V}_m^R X$ are pointed dcpos.*

Proof. Since $\bot$ is absorbing, $\tilde{\bot}$ is equal to $\bot \times \delta_x$, for any fixed $x \in X$, hence is in $\mathbf{V}_m^R X$. It is clearly least in $\mathbf{V}^R X$ and in $\mathbf{V}_m^R X$. The last claim is obvious. $\square$

Proposition 4.5 applies to the case $R = \overline{\mathbb{R}}_+$, where the bottom element is 0, and $0 \times r = r$ for every r (including $+\infty$). It also applies to the case $R = \mathbf{I}\mathbb{R}_+^\star$, where the bottom element is $[0, +\infty]$, and again $[0, +\infty] \times [a, b] = [0 \cdot_\ell a, (+\infty) \cdot_r b] = [0, +\infty]$.

Remark 4.6 There is a very similar notion of integration of interval-valued functions, yielding interval values, due to Edalat [7], which he uses to define interval-valued integrals of measurable functions. The purpose is to set up a computable framework for Lebesgue measure and integration theory. The two integrals considered in [7] and in the present paper are similar, but different in a subtle way. There are small differences, such as the fact that Edalat allows one to integrate functions with values in $\mathbf{I}\mathbb{R}$, whereas we only integrate with values in $\mathbf{I}\mathbb{R}_+^\star$, but the main difference is best illustrated by the following example. Let λ be Lebesgue measure on $[0, 1]$, and $h_n \colon [0, 1] \to \mathbf{I}\mathbb{R}_+^\star$ map every $x \in [0, 1/2^n]$ to $[0, \infty]$ and every $x \in]1/2^n, 1]$ to $[0, 0]$. The maps h_n form a chain whose supremum is the function h that maps every element of $]0, 1]$ to $[0, 0]$ and 0 to $[0, \infty]$. Using Edalat's integral, we have $\int h_n d\lambda = [0, \infty]$ for every $n \in \mathbb{N}$, but $\int h d\lambda = [0, 0]$. This shows that Edalat's integral is *not* a continuous $\mathbf{I}\mathbb{R}_+^\star$-valuation in general. We will propose a way to fix this issue in Section 7.

5

5   Monads of continuous R-valuations

We fix an Abelian d-rag R. We will see that $\mathbf{V}^R$ and $\mathbf{V}_m^R$ define strong monads on the category **Dcpo** of dcpos and Scott-continuous maps. This is essential in describing probabilistic effects, following Moggi's seminal work [22,23]. We use Manes' presentation of monads [21]: a monad $(T, \eta, \_^\dagger)$ on a category **C** is a function T mapping objects of **C** to objects of **C**, a collection of morphisms $\eta_X \colon X \to TX$, one for each object X of **C**, and called the *unit*, and for every morphism $f \colon X \to TY$, a morphism $f^\dagger \colon TX \to TY$ called the *extension* of f; those are required to satisfy the axioms:

 (i) $f^\dagger \circ \eta_X = f$;

 (ii) $\eta_X^\dagger = \mathrm{id}_{TX}$;

 (iii) $(g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger$.

Then T extends to an endofunctor, acting on morphisms through $Tf = (\eta_Y \circ f)^\dagger$.

Proposition 5.1 *The triple* $(\mathbf{V}^R, \eta, \_^\dagger)$ *is a monad on the category of dcpos and Scott-continuous maps, where* $\eta_X \colon X \to \mathbf{V}^R X$ *maps* x *to* δ_x, *and for every* $f \colon X \to \mathbf{V}^R Y$, $f^\dagger$ *is defined by* $f^\dagger(\nu)(k) \stackrel{def}{=} \nu(\lambda x \in X.f(x)(k))$ *for every* $\nu \in \mathbf{V}^R X$, *and for every* $k \in \mathcal{L}^R Y$.

Proof. Verifying that η is Scott-continuous is routine.

Let us look at $f^\dagger$. For every $k \in \mathcal{L}^R Y$, it is easy to see that $\lambda x \in X.f(x)(k)$ is Scott-continuous, because f is Scott-continuous and directed suprema are computed pointwise in $\mathbf{V}^R Y$. Hence $\nu(\lambda x \in X.f(x)(k))$ makes sense. The map $f^\dagger(\nu) \colon k \mapsto \nu(\lambda x \in X.f(x)(k))$ is also Scott-continuous, since $f(x)$ is Scott-continuous for every $x \in X$ and since ν is itself Scott-continuous. It is easy to see that $f^\dagger(\nu)$ is linear, too, because $f(x)$ is linear for every $x \in X$, and because ν is linear. Hence $f^\dagger(\nu)$ is an element of $\mathbf{V}^R Y$ for every $\nu \in \mathbf{V}^R X$. Finally, $f^\dagger$ itself is Scott-continuous, as one easily checks.

The monad equations (i), (ii) and (iii) are immediate. ☐

Fact 5.2 *The* $\mathbf{V}^R$ *functor acts on morphisms by* $\mathbf{V}^R(f)(\nu)(k) = \nu(k \circ f)$.

In integral notation, this means $\nu' \stackrel{def}{=} \mathbf{V}^R(f)(\nu)$ satisfies $\int k d\nu' = \int (k \circ f) d\nu$. This is a formula that is typical of the image measure of ν by f, where ν is a measure. We may think of $\mathbf{V}^R(f)(\nu)$ as the image of the continuous R-valuation ν by f.

We will now show that $\mathbf{V}_m^R$ defines a submonad of $\mathbf{V}^R$. To this end, we need to know more about inductive closures. A *d-closed subset* of a dcpo Z is a subset C such that the supremum of every directed family of elements of C, taken in Z, is in C. The d-closed subsets form the closed subsets of a topology called the *d-topology* [19, Section 5], and the inductive closure of a subset A coincides with its *d-closure* $cl_d(A)$, namely its closure in the d-topology.

We note that every Scott-continuous map is continuous with respect to the underlying d-topologies. This is easily checked, or see [19, Lemma 5.3]. In particular:

Fact 5.3 *For every Scott-continuous map* $f \colon \mathbf{V}^R X \to \mathbf{V}^R Y$, *for every* $A \subseteq \mathbf{V}^R X$, $f(cl_d(A)) \subseteq cl_d(f(A))$.

Lemma 5.4 *For every space* X, $\mathbf{V}_m^R X$ *is closed under addition and multiplication by elements of R, as computed in the larger space* $\mathbf{V}^R X$.

Proof. Let us deal with addition. Multiplication is similar.

For every elementary R-valuation μ, the map $f_\mu \colon \nu \in \mathbf{V}^R X \mapsto \mu + \nu$ is Scott-continuous, and maps elementary R-valuations to elementary R-valuations. By Fact 5.3 with $A \stackrel{def}{=} \mathbf{V}_f^R X$, f_μ maps all elements of $cl_d(A) = \mathbf{V}_m^R X$ to $cl_d(f_\mu(A)) \subseteq cl_d(\mathbf{V}_f^R X) = \mathbf{V}_m^R X$.

It follows that for every minimal R-valuation ν, the map $g \colon \mu \in \mathbf{V}^R X \mapsto \mu + \nu = f_\mu(\nu)$ maps elementary R-valuations to minimal R-valuations. We observe that g is also Scott-continuous. By Fact 5.3 with the same A as above, g maps all elements of $cl_d(A) = \mathbf{V}_m^R X$ to $cl_d(g(A)) \subseteq cl_d(\mathbf{V}_m^R X) = \mathbf{V}_m^R X$. Hence, for every $\nu \in \mathbf{V}_m^R X$, for every $\mu \in \mathbf{V}_m^R X$, $\mu + \nu$ is in $\mathbf{V}_m^R X$. ☐

Lemma 5.5 *For any Scott-continuous map* $f\colon X \to \mathbf{V}_m^R Y$, $f^\dagger$ *is a Scott-continuous map from* $\mathbf{V}_m^R X$ *to* $\mathbf{V}_m^R Y$.

Proof. The only challenge is to show that, for every $\nu \in \mathbf{V}_m^R X$, $f^\dagger(\nu)$ is in $\mathbf{V}_m^R Y$. Scott-continuity follows from the fact that $f^\dagger$ is Scott-continuous from $\mathbf{V}^R X$ to $\mathbf{V}^R Y$.

For every $\nu \stackrel{\text{def}}{=} \sum_{i=1}^n r_i \times \delta_{x_i}$ in $\mathbf{V}_f^R X$ ($n \geq 1$), $f^\dagger(\nu)$ is the continuous R-valuation $\sum_{i=1}^n r_i \times f(x_i)$: for every $k \in \mathcal{L}^R Y$, $f^\dagger(\nu)(k) = \nu(\lambda x \in X.f(x)(k)) = \sum_{i=1}^n r_i \times f(x_i)(k) = (\sum_{i=1}^n r_i \times f(x_i))(k)$. By Lemma 5.4, and since $f(x_i)$ is in $\mathbf{V}_m^R Y$ for each i, $f^\dagger(\nu)$ is in $\mathbf{V}_m^R Y$ as well.

Hence $f^\dagger$ maps $\mathbf{V}_f^R X$ to $\mathbf{V}_m^R Y$. Using Fact 5.3 with $A \stackrel{\text{def}}{=} \mathbf{V}_f^R X$, $f^\dagger(cl_d(A)) = f^\dagger(\mathbf{V}_m^R X)$ is included in $cl_d(f^\dagger(A)) \subseteq cl_d(\mathbf{V}_m^R Y) = \mathbf{V}_m^R Y$. $\qquad\square$

We observe that $\eta_X(x) = \delta_x$ is in $\mathbf{V}_f^R X \subseteq \mathbf{V}_m^R X$ for every dcpo X, and every $x \in X$, whence the following.

Proposition 5.6 *The triple* $(\mathbf{V}_m^R, \eta, \_^\dagger)$ *is a monad on the category of dcpos and Scott-continuous maps.*

A tensorial strength for a monad $(T, \eta, \_^\dagger)$ is a collection t of morphisms $t_{X,Y}\colon X \times TY \to T(X \times Y)$, natural in X and Y, satisfying certain coherence conditions (which we omit, see [23].) We then say that $(T, \eta, \_^\dagger, t)$ is a *strong* monad. We will satisfy ourselves with the following result. By [23, Proposition 3.4], in a category with finite products and enough points, if one can find morphisms $t_{X,Y}$ for all objects X and Y such that $t_{X,Y} \circ \langle x, \nu \rangle = T(\langle x \circ !, \mathrm{id}_Y \rangle) \circ \nu$, where $!\colon Y \to 1$ is the unique morphism from Y to the terminal object, then the collection of those morphisms is the unique tensorial strength. The category of dcpos has finite products, and has enough points, and specializing this to the $\mathbf{V}^R$ monad, we obtain the following.

Lemma 5.7 *The maps* $t_{X,Y}\colon X \times \mathbf{V}^R Y \to \mathbf{V}^R(X \times Y)$ *defined by* $t_{X,Y}(x, \nu) \stackrel{\text{def}}{=} \lambda h \in \mathcal{L}^R(X \times Y).\nu(\lambda y \in Y.h(x, y))$ *define the unique tensorial strength for the monad* $(\mathbf{V}^R, \eta, \_^\dagger)$.

Proof. The previous observation shows that we must define $t_{X,Y}$ by $t_{X,Y}(x, \nu) \stackrel{\text{def}}{=} \mathbf{V}^R(\lambda y \in Y.(x, y))(\nu)$. By Fact 5.2, the latter is equal to $\lambda h \in \mathcal{L}^R(X \times Y).\nu(h \circ \lambda y \in Y.(x, y)) = \lambda h \in \mathcal{L}^R(X \times Y).\nu(\lambda y \in Y.h(x, y))$.

It is enough to check that $t_{X,Y}$ is Scott-continuous. This follows from the fact that application (of ν to $\lambda y \in Y.h(x, y))$) is Scott-continuous. $\qquad\square$

In integral notation, $t_{X,Y}(x, \nu)$ is the continuous R-valuation ν' such that $\int h d\nu' = \int h(x, \_) d\nu$ for every $h \in \mathcal{L}^R(X \times Y)$.

For every $\nu \in \mathbf{V}_m^R Y$, for every $x \in X$, $t_{X,Y}(x, \nu)$ is equal to $\mathbf{V}^R(\lambda y \in Y.(x, y))(\nu) = (\eta \circ \lambda y \in Y.(x, y))^\dagger(\nu)$. By Lemma 5.5, this is an element of $\mathbf{V}_m^R(X \times Y)$. It follows:

Proposition 5.8 $(\mathbf{V}^R, \eta, \_^\dagger, t)$ *and* $(\mathbf{V}_m^R, \eta, \_^\dagger, t)$ *are strong monads on* **Dcpo**.

We now show that $\mathbf{V}_m^R$ is a *commutative* monad. The corresponding result is unknown for $\mathbf{V}^R$, even when $R = \overline{\mathbb{R}}_+$.

Given a tensorial strength t, there is a dual tensorial strength t', where $t'_{X,Y}\colon TX \times Y \to T(X \times Y)$. Here $t'_{X,Y}(\mu, y) = \lambda h \in \mathcal{L}^R(X \times Y).\mu(\lambda x \in X.h(x, y))$. We can then define two morphisms from $TX \times TY$ to $T(X \times Y)$, namely $t'^\dagger_{X,Y} \circ t_{TX,Y}$ and $t^\dagger_{X,Y} \circ t'_{X,TY}$. The monad T is *commutative* when they coincide.

Lemma 5.9 *Two morphisms* $f, g\colon X \to Y$ *in* **Dcpo** *that coincide on* $A \subseteq X$ *also coincide on* $cl_d(A)$.

Proof. Let $B \stackrel{\text{def}}{=} \{x \in X \mid f(x) = g(x)\}$. Since f and g preserve directed suprema, B is d-closed. By assumption, A is included in B, so B also contains $cl_d(A)$. $\qquad\square$

Proposition 5.10 *Let* X, Y *be two dcpos. The maps* $t^\dagger_{X,Y} \circ t'_{X,TY}$ *and* $t'^\dagger_{X,Y} \circ t_{TX,Y}$ *coincide on those pairs* $(\mu, \nu) \in \mathbf{V}^R X \times \mathbf{V}^R Y$ *such that* $\mu \in \mathbf{V}_m^R X$ *or* $\nu \in \mathbf{V}_m^R Y$.

Proof. We prove the claim when $\mu \in \mathbf{V}_m^R X$. The case where $\nu \in \mathbf{V}_m^R Y$ is symmetric.

In the sequel, h ranges over $\mathcal{L}^R(X \times \mathbf{V}^R Y)$, k over $\mathcal{L}^R(X \times Y)$, x over X, y over Y, and ν over $\mathbf{V}^R Y$. For all $\mu \in \mathbf{V}^R X$ and $\nu \in \mathbf{V}^R Y$, we verify that:

$$(t_{X,Y}^\dagger \circ t'_{X,TY})(\mu, \nu) = \lambda k.\mu(\lambda x.\nu(\lambda y.k(x,y)))$$
$$(t'^\dagger_{X,Y} \circ t_{TX,Y})(\mu, \nu) = \lambda k.\nu(\lambda y.\mu(\lambda x.k(x,y))).$$

Those two quantities are equal when μ is an elementary R-valuation $\sum_{i=1}^m r_i \times \delta_{x_i}$ ($m \geq 1$), since the first one is equal to $\lambda k.\sum_{i=1}^m r_i \times \nu(\lambda y.k(x_i, y))$, the second one is equal to $\lambda k.\nu(\lambda y.\sum_{i=1}^m r_i \times k(x_i, y))$, and since ν is linear.

For fixed $\nu \in \mathbf{V}^R Y$, the maps $f\colon \mu \in \mathbf{V}^R X \mapsto (t_{X,Y}^\dagger \circ t'_{X,TY})(\mu, \nu)$ and $g\colon \mu \in \mathbf{V}^R X \mapsto (t'^\dagger_{X,Y} \circ t_{TX,Y})(\mu, \nu)$ therefore coincide on $\mathbf{V}_f^R X$. They are both Scott-continuous, since $t_{X,Y}^\dagger \circ t'_{X,TY}$ and $t'^\dagger_{X,Y} \circ t_{TX,Y}$ are. By Lemma 5.9, they must coincide on the d-closure of $\mathbf{V}_f^R X$, which is $\mathbf{V}_m^R X$ by definition. $\square$

Corollary 5.11 $(\mathbf{V}_m^R, \eta, \_^\dagger, t)$ *is a commutative monad.*

The equality of Proposition 5.10 is that, for every $\mu \in \mathbf{V}_m^R X$ and for every $\nu \in \mathbf{V}_m^R Y$, for every $k \in \mathcal{L}^R(X \times Y)$, $\mu(\lambda x \in X.\nu(\lambda y \in Y.k(x,y))) = \nu(\lambda y \in Y.\mu(\lambda x \in X.k(x,y)))$. In integral notation,

$$\int_{x \in X} \left(\int_{y \in Y} k(x,y) d\nu \right) d\mu = \int_{y \in Y} \left(\int_{x \in X} k(x,y) d\mu \right) d\nu,$$

which we recognize as the integral permutation property, obtained in the classical measure-theoretic case as a consequence of Fubini's theorem.

Fubini's theorem is more general, and states the existence of a product measure. A similar fact follows from the above results, as noticed by Kock [20]. We write $\otimes$ for the morphism $t'^\dagger_{X,Y} \circ t_{TX,Y}$ and $t_{X,Y}^\dagger \circ t'_{X,TY}$ from $\mathbf{V}_m^R X \times \mathbf{V}_m^R Y$ to $\mathbf{V}_m^R(X \times Y)$, as with any commutative monad [20, Section 5]. Then, for all $\mu \in \mathbf{V}_m^R X$ and $\nu \in \mathbf{V}_m^R Y$, $\otimes(\mu, \nu)$, which we prefer to write as $\mu \otimes \nu$, is in $\mathbf{V}_m^R(X \times Y)$, and by definition $(\mu \otimes \nu)(k) = \mu(\lambda x \in X.\nu(\lambda y \in Y.k(x,y))) = \nu(\lambda y \in Y.\mu(\lambda x \in X.k(x,y)))$. In integral notation, we obtain the following form of Fubini's theorem:

$$\int_{(x,y) \in X \times Y} k(x,y) d(\mu \otimes \nu) = \int_{x \in X} \left(\int_{y \in Y} k(x,y) d\nu \right) d\mu = \int_{y \in Y} \left(\int_{x \in X} k(x,y) d\mu \right) d\nu,$$

for all $\mu \in \mathbf{V}_m^R X$, $\nu \in \mathbf{V}_m^R Y$, and $k \in \mathcal{L}^R(X \times Y)$. As an additional benefit, we obtain (for free!) that the map $\otimes\colon (\mu, \nu) \mapsto \mu \otimes \nu$ is Scott-continuous.

Remark 5.12 A similar Fubini-like theorem was already obtained by Jones [17] for arbitrary (subprobability) continuous valuations, but in the setting of continuous dcpos only. Whether the Fubini-like formula above holds for every pair of continuous valuations μ and ν on arbitrary dcpos is unknown. We note that the problem would be easily solved if all continuous valuations were minimal, but that is not the case, as is shown in the paper [13].

6   Continuous R-valuations and measures I: A brief viewpoint

We look at the special cases of continuous R-valuations when R is $\overline{\mathbb{R}}_+$ or $\mathbf{I}\mathbb{R}_+^\star$, and we investigate their relations to measures.

When $R = \overline{\mathbb{R}}_+$, this is simple: as noticed in Remark 4.2, we can equate continuous R-valuations with continuous valuations. Next, continuous valuations and measures are pretty much the same thing on ω-continuous dcpos, namely on continuous dcpos with a countable basis. This holds more generally on de Brecht's quasi-Polish spaces [5], a class of spaces that contains not only the ω-continuous dcpos from domain theory but also the Polish spaces from topological measure theory. One can see this as follows. In

one direction, every measure μ on a hereditarily Lindelöf space X is τ-smooth [2, Theorem 3.1], meaning that its restriction to the lattice of open subsets of X is a continuous valuation. A hereditarily Lindelöf space is a space whose subspaces are all Lindelöf, or equivalently a space in which every family of open sets contains a countable subfamily with the same union. Every second-countable space is hereditarily Lindelöf, and that includes all quasi-Polish spaces. In the other direction, every continuous valuation on an LCS-complete space extends to a Borel measure [6, Theorem 1.1]. An *LCS-complete* space is a space that is homeomorphic to a G_δ subset of a locally compact sober space. Every quasi-Polish space is LCS-complete; in fact, the quasi-Polish spaces are exactly the second-countable LCS-complete spaces [6, Theorem 9.5].

Remark 6.1 A continuous valuation μ on an LCS-complete space X may extend to more than one Borel measure. However, the extension is unique if μ is σ-*finite*, namely if there is a monotone sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n \subseteq \cdots$ of open subsets of X whose union is the whole of X, and such that $\mu(U_n) < +\infty$ for every $n \in \mathbb{N}$. Indeed, any extension of μ will be σ-finite in the usual sense. We conclude since any two σ-finite measures that agree on all open sets (which form a π-system) must agree on the Borel σ-algebra.

We now look in detail at the more complex case $R = \mathbb{IR}_+^\star$. We use the following notation. Given any element $\mathbf{x}$ of $\mathbb{IR}_+^\star$ or of $\mathbb{IR}^\star$, we write x^- and x^+ for its endpoints, viz., $\mathbf{x} = [x^-, x^+]$. Every map $h \colon X \to \mathbb{IR}^\star$ defines two maps $h^-, h^+ \colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $h^-(x) \overset{\text{def}}{=} h(x)^-$ and $h^+(x) \overset{\text{def}}{=} h(x)^+$. Given two maps $f, g \colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that $f \leq g$, we write $[f, g]$ for the function that maps x to $[f(x), g(x)]$. Note that, given any map f in $\mathcal{L}X$, the map $[f, +\infty.\mathbf{1}] \colon x \mapsto [f(x), +\infty]$ is in $\mathcal{L}^{\mathbb{IR}_+^\star}X$. (We will write $r.\mathbf{1}$ for the constant function with value r, in order to distinguish it from the scalar value r.) Given any $\mathbb{IR}_+^\star$-continuous valuation F on a space X, we also define $F^-(h)$ as $F(h)^-$ and $F^+(h)$ as $F(h)^+$, for every $h \in \mathcal{L}^{\mathbb{IR}_+^\star}X$.

Lemma 6.2 (The view from the left I) *Let X be any topological space. For every continuous $\mathbb{IR}_+^\star$-valuation F on X, for every $h \in \mathcal{L}^{\mathbb{IR}_+^\star}X$, $F^-(h)$ only depends on h^-, not on h^+. Moreover, there is a unique continuous valuation ν_F on X such that, for every $h \in \mathcal{L}^{\mathbb{IR}_+^\star}X$,*

$$F^-(h) = \int_X h^- d\nu_F.$$

Proof. For the first part, it suffices to show that $F^-(h) = F^-([h^-, +\infty.\mathbf{1}])$. We note that the bottom element $[0, +\infty]$ of $\mathbb{IR}_+^\star$ is multiplicatively absorbing: for every $\mathbf{x} \in \mathbb{IR}_+^\star$, $[0, +\infty] \times \mathbf{x} = [0, +\infty]$. It follows that

$$F([0.\mathbf{1}, +\infty.\mathbf{1}]) = F([0, +\infty] \times [0.\mathbf{1}, +\infty.\mathbf{1}])$$
$$= [0, +\infty] \times F([0.\mathbf{1}, +\infty.\mathbf{1}]) = [0, +\infty].$$

Next, $[0, +\infty]$ satisfies the following partial absorption law for addition: $\mathbf{x} \in \mathbb{IR}_+^\star$, $[0, +\infty] + \mathbf{x} = [x^-, +\infty]$. Therefore,

$$\begin{aligned} F([h^-, +\infty.\mathbf{1}]) &= F(h + [0.\mathbf{1}, +\infty.\mathbf{1}]) \\ &= F(h) + F([0.\mathbf{1}, +\infty.\mathbf{1}]) \\ &= F(h) + [0, +\infty] \qquad\qquad \text{by our previous result} \\ &= [F^-(h), +\infty]. \end{aligned}$$

It follows that $F^-(h) = F^-([h^-, +\infty.\mathbf{1}])$, and the right-hand side does not depend on h^+.

In order to show the second part of the lemma, it suffices to observe that the map $f \mapsto F^-([f, +\infty.\mathbf{1}])$ is linear and Scott-continuous, and is therefore the integral functional of a unique continuous valuation ν_F. $\qquad\square$

It follows that, for every $f \in \mathcal{L}X$, $\int_X f d\nu_F = F^-([f, +\infty.\mathbf{1}])$. In particular, for every open subset U of X, $\nu_F(U) = F^-([\chi_U, +\infty.\mathbf{1}])$.

There are many ways in which we can reconstruct a continuous $\mathbb{IR}_+^\star$-valuation from a continuous valuation, and here is the simplest of all.

Lemma 6.3 (The view from the left II) *Let X be any topological space. For every continuous valuation ν on X, there is a smallest continuous $\mathbb{IR}_+^\star$-valuation F such that $\nu_F = \nu$. For every $h \in \mathcal{L}^{\mathbb{IR}_+^\star}X$,*

$$F(h) \stackrel{def}{=} \left[\int_X h^- d\nu, +\infty \right].$$

For the continuous $\mathbb{IR}_+^\star$-valuation just given, the view from the right, namely $F^+(h)$, is the constant $+\infty$, for every integrand h, including for the constant zero map. This cannot be the integral of h with respect to any measure, since the integral of the zero map is always zero, with respect to any continuous valuation or measure.

One possible view of continuous $\mathbb{IR}_+^\star$-valuations F is that of the specification of some unknown measure. F^- gives a continuous valuation that is a lower bound on that measure, while F^+ measures how precise that specification is. In this setting, the continuous $\mathbb{IR}_+^\star$-valuation F built in Lemma 6.3 is the least precise specification for ν.

On more special topological spaces, we will see that every measure has a much more precise specification, and that it is minimal.

7 Continuous R-valuations and measures II: Measures as continuous $\mathbb{IR}_+^\star$-valuations

We will see that every non-zero, bounded τ-smooth measure μ on a coherent topological space X gives rise to a continuous $\mathbb{IR}_+^\star$-valuation in a natural way. (A measure μ on X is *bounded* if $\mu(X) < \infty$, and we recall that it is τ-smooth if and only if it restricts to a continuous valuation on $\mathcal{O}X$.) As a first step, we need to define integrals of functions with values in $\overline{\mathbb{R}}_+$, not just $\mathbb{R}_+$, as is done classically.

More precisely, given a measure μ on a topological space X (with its Borel σ-algebra), we can define the Lebesgue integral $\int_{x \in X} f(x) d\mu$ of any measurable map $f \colon X \to \mathbb{R}_+$. We extend this definition to measurable maps f from X to $\overline{\mathbb{R}}_+$. Just as with multiplication in rags, this comes in two flavors.

Perhaps the most natural extension is:

$$\int_{x \in X}^- f(x) d\mu \stackrel{def}{=} \sup_{r \in \mathbb{R}_+}^\uparrow \int_{x \in X} \min(f(x), r) d\mu. \tag{1}$$

It is known that the Lebesgue integral, as used on the right of (1) can be defined through the following, so-called Choquet formula [4, Chapter VII, Section 48.1, p. 265]:

$$\int_{x \in X} f(x) d\mu = \int_0^\infty \mu(f^{-1}(]t, \infty])) dt \tag{2}$$

where the integral on the right is now an indefinite Riemann integral. As a consequence, and since $(\min(f(\_), r)^{-1}(]t, \infty])$ is empty for every $t \geq r$, and equal to $f^{-1}(]t, \infty])$ for every $t < r$, we can rewrite (1) as:

$$\int_{x \in X}^- f(x) d\mu = \sup_{r \in \mathbb{R}_+}^\uparrow \int_0^r \mu(f^{-1}(]t, \infty])) dt. \tag{3}$$

We observe that this lower integral is linear and ω-continuous (by the monotone convergence theorem). It also commutes with the product structure of the d-rag $\overline{\mathbb{R}}_+$. We also note the following change of variable formula, for future reference:

$$\int_{y \in Y}^- f(y) dj[\mu] = \int_{x \in X}^- f(j(x)) d\mu, \tag{4}$$

for every measurable map $j\colon X \to Y$, for every measurable map $f\colon Y \to \overline{\mathbb{R}}_+$, for every measure μ on X, and where $j[\mu]$ is the *image measure*, defined by $j[\mu](E) \overset{\text{def}}{=} \mu(j^{-1}(E))$. This is an obvious consequence of (3).

A function $f\colon X \to \overline{\mathbb{R}}_+$ is lower semicontinuous if and only if it is continuous from X to $\overline{\mathbb{R}}_+$ with the Scott topology; equivalently, for every $r \in \mathbb{R}_+ \setminus \{0\}$, $f^{-1}(]r, \infty])$ is open in X. Every lower semicontinuous function is measurable.

Lemma 7.1 *The lower integral* (1) *is:*

(i) *additive: for all measurable maps f, g from X to $\overline{\mathbb{R}}_+$, $\int^-_{x \in X}(f(x) + g(x))d\mu = \int^-_{x \in X} f(x)d\mu + \int^-_{x \in X} g(x)d\mu$;*

(ii) *$\cdot_\ell$-homogeneous: for every measurable map $f\colon X \to \overline{\mathbb{R}}_+$, for every $a \in \overline{\mathbb{R}}_+$, $a \cdot_\ell \int^-_{x \in X} f(x)d\mu = \int^-_{x \in X}(a \cdot_\ell f(x))d\mu$;*

(iii) *ω-continuous: for every monotonic sequence $(f_n)_{n \in \mathbb{N}}$ of measurable maps from X to $\overline{\mathbb{R}}_+$, $\int^-_{x \in X} \sup^\uparrow_{n \in \mathbb{N}} f_n(x)d\mu = \sup^\uparrow_{n \in \mathbb{N}} \int^-_{x \in X} f_n(x)d\mu$;*

(iv) *Scott-continuous on lower semicontinuous maps, provided that μ is τ-smooth: for every directed family $(f_i)_{i \in I}$ of lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$, $\int^-_{x \in X} \sup^\uparrow_{i \in I} f_i(x)d\mu = \sup^\uparrow_{i \in I} \int^-_{x \in X} f_i(x)d\mu$.*

Proof. 1. This follows from the additivity of Lebesgue integral of $\mathbb{R}_+$-valued functions and the inequalities $\min(f(x) + g(x), r) \le \min(f(x), r) + \min(g(x), r) \le \min(f(x) + g(x), 2r)$.

2. For every $a \in \mathbb{R}_+ \setminus \{0\}$,

$$a \cdot_\ell \int^-_{x \in X} f(x)d\mu = \sup^\uparrow_{r \in \mathbb{R}_+} \int_{x \in X} a \min(f(x), r)d\mu$$
$$= \sup^\uparrow_{r' \in \mathbb{R}_+} \int_{x \in X} \min(a \cdot_\ell f(x), r')d\mu \qquad \text{by letting } r' \overset{\text{def}}{=} ar$$
$$= \int^-_{x \in X}(a \cdot_\ell f(x))d\mu.$$

When $a = 0$, $a \cdot_\ell \int^-_{x \in X} f(x)d\mu$ is equal to 0 by definition, and $\int^-_{x \in X}(a \cdot_\ell f(x))d\mu = \int_{x \in X} 0 d\mu = 0$.

The new, key case is when $a = \infty$. We split this into two subcases. If $\mu(f^{-1}(]0, \infty])) = 0$, namely if f is μ-a.e. zero, then $\min(f(\_), r)$ and $\min(\infty \cdot_\ell f(\_), r)$ are also μ-a.e. zero, so $\infty \cdot_\ell \int^-_{x \in X} f(x)d\mu$ and $\int^-_{x \in X}(\infty \cdot_\ell f(x))d\mu$ are both equal to 0. If $\mu(f^{-1}(]0, \infty])) > 0$, then $\mu(f^{-1}(]r, \infty])) > 0$ for some $r \in \mathbb{R}_+ \setminus \{0\}$, since $\mu(f^{-1}(]0, \infty])) = \sup^\uparrow_{q \in \mathbb{Q}_+ \setminus \{0\}} \mu(f^{-1}(]q, \infty]))$. It follows that $\int^-_{x \in X} f(x)d\mu \ge r\mu(f^{-1}(]r, \infty])) > 0$, so $\infty \cdot_\ell \int^-_{x \in X} f(x)d\mu = \infty$, while $\int^-_{x \in X} \infty \cdot_\ell f(x)d\mu \ge \int_{x \in X} \min(\infty \cdot_\ell f(x), r)d\mu \ge r\mu(f^{-1}(]0, \infty]))$ for every $r \in \mathbb{R}_+ \setminus \{0\}$, so that $\int^-_{x \in X} \infty \cdot_\ell f(x)d\mu = \infty$ as well.

3. We use the monotone convergence theorem, and the fact that $\sup^\uparrow_{n \in \mathbb{N}} \min(f_n(x), r) = \min(\sup^\uparrow_{n \in \mathbb{N}} f_n(x), r)$ for all $x \in X$ and $r \in \mathbb{R}_+$.

4. Since Riemann integration of non-increasing maps from $\overline{\mathbb{R}}_+$ to $\overline{\mathbb{R}}_+$ is Scott-continuous (see for example Lemma 4.2 in [26]), it follows from (3) that the lower integral $\int_{x \in X} f(x)d\mu$ is Scott-continuous in the lower semicontinuous map f, provided that μ is τ-smooth. $\square$

We also consider the following, upper integral. This will really only make sense when the integrated function f is upper semicontinuous, namely when for every $r \in \mathbb{R}_+ \setminus \{0\}$, $f^{-1}([0, r[)$ is open in X; and when the measure μ is non-zero ($\mu(X) \neq 0$), and τ-smooth.

A *support* of a measure μ on X is any set E such that, for all measurable subsets A and B of X such that $A \cap E = B \cap E$, $\mu(A) = \mu(B)$. When E is itself measurable, this is equivalent to requiring $\mu(E) = \mu(X)$, and when μ is additionally bounded (i.e., $\mu(X) < \infty$), this is equivalent to $\mu(X \setminus E) = 0$. We sometimes say that μ is *supported on E* to mean that E is a support of μ.

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For every τ-smooth measure μ on X, the intersection of all closed supports of μ is again a closed support of μ: this smallest closed support will be denote as supp μ. But beware that there might be smaller (non-closed) supports. For example, supp (δ_x) is equal to the closure $\downarrow x$ of the point x, but $\{x\}$ is a smaller (non-closed) support. Note that $\{x\}$ is the intersection of the compact saturated set $\uparrow x$, which happens to be a support of $\mu = \delta_x$, with supp (μ).

In general, not all compact saturated sets Q are measurable, so we will restrict to *measurable* compact saturated subsets in the sequel. The intersection of two supports E and E' may also fail to be a support, but if one of them (say E') is measurable, then $E \cap E'$ is also a support. (Indeed, let A, B be measurable such that $A \cap (E \cap E') = B \cap (E \cap E')$. Then $(A \cap E') \cap E = (B \cap E') \cap E$, and since E is a support of μ, $\mu(A \cap E') = \mu(B \cap E')$. Since E' is a support of μ, and since A and $A \cap E'$ have the same intersection with E', $\mu(A \cap E') = \mu(A)$, and similarly $\mu(B \cap E') = \mu(B)$. Therefore $\mu(A) = \mu(B)$.)

For every τ-smooth measure μ on X, we say that a measurable map $f \colon X \to \overline{\mathbb{R}}_+$ is μ-*bounded* if and only if there is a measurable compact saturated support Q of μ such that f is bounded on $Q \cap \operatorname{supp} \mu$, namely if $\sup_{x \in Q \cap \operatorname{supp} \mu} f(x) < \infty$. We will also say that Q is a *witness* of μ-boundedness of f, or that f is μ-bounded, *witnessed by* Q, in that case. We say that f is μ-*unbounded* if it is not μ-bounded.

Lemma 7.2 *For every non-zero τ-smooth measure μ on a topological space X, for every compact saturated support Q of μ, $Q \cap \operatorname{supp} \mu$ is non-empty.*

Proof. Otherwise, supp μ and the empty set have the same intersection with Q, and since Q is a support of μ, we would have $\mu(\operatorname{supp} \mu) = \mu(\emptyset) = 0$. Since supp μ is a measurable support of μ, $\mu(\operatorname{supp} \mu) = \mu(X)$, and therefore we would have $\mu(X) = 0$, contradicting the fact that μ is non-zero. $\qquad\square$

Lemma 7.3 *For every non-zero τ-smooth measure μ on a topological space X, and for every upper semicontinuous map $f \colon X \to \overline{\mathbb{R}}_+$, for every compact saturated support Q of μ, there is a point $x \in Q \cap \operatorname{supp} \mu$ such that $f(x) = \sup_{x \in Q \cap \operatorname{supp} \mu} f(x)$.*

Proof. Every upper semicontinuous $\overline{\mathbb{R}}_+$-valued function f reaches its maximum on any non-empty compact set K. Here is a quick proof: let $a \overset{\text{def}}{=} \sup_{x \in K} f(x)$, and assume that $f(x) < a$ for every $x \in K$. The open sets $f^{-1}([0, r[)$ with $r \in [0, a[$ form an open cover of K. We extract a finite subcover $f^{-1}([0, r[)$, where r ranges over some finite set A of numbers strictly below a. This implies that, for every $x \in K$, $f(x) < r$ for some $r \in A$, so that $a = \sup_{x \in K} f(x) < \max A < a$, a contradiction.

We now apply this to $K \overset{\text{def}}{=} Q \cap \operatorname{supp} \mu$, which is non-empty by Lemma 7.2. $\qquad\square$

Corollary 7.4 *For every non-zero τ-smooth measure μ on a topological space X, for every upper semicontinuous map $f \colon X \to \overline{\mathbb{R}}_+$, f is μ-unbounded if and only if for every measurable compact saturated support Q of μ, there is a point $x \in Q \cap \operatorname{supp} \mu$ such that $f(x) = \infty$.* $\qquad\square$

This being done, for a τ-smooth measure μ and an upper semicontinuous map $f \colon X \to \overline{\mathbb{R}}_+$, we define:

$$\int_{x \in X}^{+} f(x)d\mu \overset{\text{def}}{=} \begin{cases} \int_{x \in X}^{-} f(x)d\mu & \text{if } f \text{ is } \mu\text{-bounded} \\ \infty & \text{otherwise.} \end{cases} \tag{5}$$

We say that a topological space is *coherent* if and only if the intersection of any two compact saturated subets is compact (and saturated).

Lemma 7.5 *Let μ be a non-zero τ-smooth measure on a topological space X. The upper integral (5) is:*

(i) *additive if X is coherent: for all upper semicontinuous maps f, g from X to $\overline{\mathbb{R}}_+$, $\int_{x \in X}^{+}(f(x) + g(x))d\mu = \int_{x \in X}^{+} f(x)d\mu + \int_{x \in X}^{+} g(x)d\mu$;*

(ii) *$\cdot_r$-homogeneous: for every upper semicontinuous map $f \colon X \to \overline{\mathbb{R}}_+$, for every $a \in \overline{\mathbb{R}}_+$, $a \cdot_r \int_{x \in X}^{+} f(x)d\mu = \int_{x \in X}^{+}(a \cdot_r f(x))d\mu$;*

(iii) *Scott-cocontinuous if μ is also bounded: for every filtered family $(f_i)_{i \in I}$ of upper semicontinuous maps from X to $\overline{\mathbb{R}}_+$, $\int_{x \in X}^{+} \inf_{i \in I}^{\downarrow} f_i(x)d\mu = \inf_{i \in I}^{\downarrow} \int_{x \in X}^{+} f_i(x)d\mu$;*

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(iv) *above the lower integral: for every measurable map $g\colon X \to \overline{\mathbb{R}}_+$, for every upper semicontinuous map $f\colon X \to \overline{\mathbb{R}}_+$ such that $g \le f$ on $E \cap \operatorname{supp}\mu$, where E is any measurable support of μ, $\int_{x\in X}^{-} g(x)d\mu \le \int_{x\in X}^{+} f(x)d\mu$;*

(v) *for every μ-bounded upper semicontinuous map $g\colon X \to \overline{\mathbb{R}}_+$, witnessed by Q, $\int_{x\in X}^{-} g(x)d\mu = \int_{x\in X}^{+} g(x)d\mu$ is the usual Lebesgue integral $\int_{x\in X} g(x)\mathbf{1}_{Q\cap \operatorname{supp}\mu}(x)d\mu$;*

Proof. We prove item 5 first. When g is μ-bounded, witnessed by Q, we can define a new measurable map $g.\mathbf{1}_{Q\cap\operatorname{supp}\mu}$, which maps every $x \in Q \cap \operatorname{supp}\mu$ to $g(x)$, and all other points to 0. Then $g.\mathbf{1}_{Q\cap\operatorname{supp}\mu}$ is bounded, and coincides with g on $Q \cap \operatorname{supp}\mu$. Since the latter is a support of μ, it is an easy exercise, using (3), to show that $\int_{x\in X}^{-} g(x)d\mu$ is equal to $\int_{x\in X}^{-} g(x).\mathbf{1}_{Q\cap\operatorname{supp}\mu}(x)d\mu$, which is the ordinary Lebesgue integral $\int_{x\in X} g(x).\mathbf{1}_{Q\cap\operatorname{supp}\mu}(x)d\mu$.

1. If f and g are both μ-bounded, witnessed respectively by Q and Q', then so is $f + g$, witnessed by $Q \cap Q'$. The latter is measurable, and compact saturated since X is coherent. It is also a support of μ, since Q (or Q') is measurable. The claim then follows from Lemma 7.1, item 1.

If, say, f is not μ-bounded, then for every measurable compact saturated support Q of μ, there is a point $x \in Q \cap \operatorname{supp}\mu$ such that $f(x) = \infty$ by Corollary 7.4. Then, $f(x) + g(x)$ is also equal to ∞, showing that $f + g$ is not μ-bounded either. In particular, $\int_{x\in X}^{+}(f(x) + g(x))d\mu$ and $\int_{x\in X}^{+} f(x)d\mu + \int_{x\in X}^{+} g(x)d\mu$ are both equal to ∞.

2. If f is μ-bounded and $a \neq \infty$, then $a \cdot_r f$ is also μ-bounded: for every measurable compact saturated support Q of μ, f and therefore $a \cdot_r f$ is bounded on $Q \cap \operatorname{supp}\mu$. Then the claim follows from Lemma 7.1, item 2, and the fact that $\cdot_\ell$ and $\cdot_r$ both coincide with the ordinary product on $\mathbb{R}_+$.

If $a = \infty$, then by definition $\infty \cdot_r \int_{x\in X}^{+} f(x)d\mu = \infty$, since ∞ is absorbing for $\cdot_r$; and $\int_{x\in X}^{+}(\infty \cdot_r f(x))d\mu = \int_{x\in X}^{+} \infty d\mu = \infty$. The latter equality follows from the fact that the constant map ∞ is not μ-bounded; indeed, for every measurable compact saturated support Q of μ, $Q \cap \operatorname{supp}\mu$ is non-empty by Lemma 7.2, so that ∞ is not bounded on that set.

If f is not μ-bounded but $a \in \mathbb{R}_+$, then for every measurable compact saturated support Q of μ, there is a point $x \in Q \cap \operatorname{supp}\mu$ such that $f(x) = \infty$ by Corollary 7.4. Then $a \cdot_r f(x) = \infty$ as well. This shows that $a \cdot_r f$ is not μ-bounded either. It follows that $\int_{x\in X}^{+}(a \cdot_r f(x))d\mu = \infty$, while $a \cdot_r \int_{x\in X}^{+} f(x)d\mu = a \cdot_r \infty = \infty$.

3. First, the pointwise infimum $f \stackrel{\text{def}}{=} \inf^{\downarrow}_{i\in I} f_i$ of upper semicontinuous maps f_i's is upper semicontinuous. Let us write $i \preceq j$ if and only if $f_i \le f_j$.

If f_{i_0} is μ-bounded for some $i_0 \in I$, then $f_i \le f_{i_0}$ is also μ-bounded for every $i \preceq i_0$, and witnessed by the same measurable compact saturated set Q. Similarly, f is also μ-bounded, witnessed by Q. We let r be an upper bound of f_{i_0} on $Q \cap \operatorname{supp}\mu$. Then, using item 5,

$$\int_{x\in X}^{+} f(x)d\mu = \int_{x\in X} f(x)\mathbf{1}_{Q\cap\operatorname{supp}\mu}(x)d\mu$$
$$= r\mu(X) - \int_{x\in X}(r - f(x))\mathbf{1}_{Q\cap\operatorname{supp}\mu}(x)d\mu.$$

Indeed, the map $(r - f(\_))\mathbf{1}_{Q\cap\operatorname{supp}\mu}$ also takes its values in $\mathbb{R}_+$, and the sum of $\int_{x\in X} f(x)\mathbf{1}_{Q\cap\operatorname{supp}\mu}(x)d\mu$ and of $\int_{x\in X}(r - f(x))\mathbf{1}_{Q\cap\operatorname{supp}\mu}(x)d\mu$ is equal to $\int_{x\in X} r\mathbf{1}_{Q\cap\operatorname{supp}\mu}(x)d\mu = r\mu(Q \cap \operatorname{supp}\mu) = r\mu(X)$, since $Q \cap \operatorname{supp}\mu$ is a measurable support of μ. Since integration of lower semicontinuous maps with respect to

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a τ-smooth measure is Scott-continuous, as in Lemma 7.1, item 4, we obtain:

$$\int_{x\in X}^{+} f(x)d\mu = r\mu(X) - \sup_{i\preceq i_0}^{\uparrow} \int_{x\in X} (r - f_i(x))\mathbf{1}_{Q\cap\mathrm{supp}\,\mu}(x)d\mu$$

$$= \inf_{i\preceq i_0}^{\downarrow} \int_{x\in X} f_i(x)\mathbf{1}_{Q\cap\mathrm{supp}\,\mu}(x)d\mu$$

$$= \inf_{i\preceq i_0}^{\downarrow} \int_{x\in X}^{+} f_i(x)d\mu = \inf_{i\in I}^{\downarrow} \int_{x\in X}^{+} f_i(x)d\mu.$$

If no f_i is μ-bounded, then f cannot be μ-bounded either, as we now claim. If f is μ-bounded, witnessed by Q, then, let $r \in \mathbb{R}_+$ be such that for every $x \in Q \cap \mathrm{supp}\,\mu$, $f(x) < r$. Since $f = \inf_{i\in I}^{\downarrow} f_i$, every point x of $Q \cap \mathrm{supp}\,\mu$ is in the open set $f_i^{-1}([0, r[)$ for some $i \in I$. The family $(f_i^{-1}([0, r[))_{i\in I}$ is then an open cover of $Q \cap \mathrm{supp}\,\mu$. The intersection of a compact set and of a closed set is compact, so $Q \cap \mathrm{supp}\,\mu$ is compact, and therefore $(f_i^{-1}([0, r[))_{i\in I}$ has a finite subcover. Since $(f_i^{-1}([0, r[))_{i\in I}$ is a directed family, we can assume that this subcover consists of just one open set $f_i^{-1}([0, r[)$. But that implies that f_i is bounded on $Q \cap \mathrm{supp}\,\mu$, hence μ-bounded, a contradiction.

Hence we have proved that f is not μ-bounded, so $\int_{x\in X}^{+} f(x)d\mu = \infty$, which is then vacuously equal to $\inf_{i\in I}^{\downarrow} \int_{x\in X}^{+} f_i(x)d\mu$.

4. When f is μ-bounded, witnessed by Q, $\int_{x\in X}^{+} f(x)d\mu$ is equal to $\int_{x\in X}^{-} f(x)d\mu$, hence to the ordinary integral $\int_{x\in X} f(x).\mathbf{1}_{Q\cap\mathrm{supp}\,\mu}(x)d\mu$ by item 5. Since E is a measurable support of μ, $E \cap Q \cap \mathrm{supp}\,\mu$ is also a (measurable) support of μ, so the latter is also equal to $\int_{x\in X} f(x).\mathbf{1}_{E\cap Q\cap\mathrm{supp}\,\mu}(x)d\mu$. Since g is below f on $E \cap \mathrm{supp}\,\mu$, $g \cdot \mathbf{1}_{E\cap Q\cap\mathrm{supp}\,\mu}$ is (bounded and) below $f \cdot \mathbf{1}_{E\cap Q\cap\mathrm{supp}\,\mu}$, so $\int_{x\in X}^{+} f(x)d\mu = \int_{x\in X} f(x).\mathbf{1}_{E\cap Q\cap\mathrm{supp}\,\mu}(x)d\mu$ is larger than or equal to $\int_{x\in X} g(x).\mathbf{1}_{E\cap Q\cap\mathrm{supp}\,\mu}(x)d\mu$, and the latter is equal to $\int_{x\in X}^{-} g(x)d\mu$ by a similar argument.

If f is not μ-bounded, then $\int_{x\in X}^{+} f(x)d\mu = \infty$, and the claim is trivial. $\qquad\square$

We let $R \overset{\text{def}}{=} \mathbf{I}\mathbb{R}_+^{\star}$, and we fix a topological space X. For every $h \in \mathcal{L}^R X$, for every $x \in X$, $h(x)$ is an interval $[h^-(x), h^+(x)]$. The function h^- is lower semicontinuous. Indeed, for every $r \in \mathbb{R}_+ \setminus \{0\}$, $(h^-)^{-1}(]r, \infty]) = h^{-1}(\uparrow[r, \infty])$: for every $[a, b] \in \mathbf{I}\mathbb{R}_+^{\star}$, $[r, \infty] \ll [a, b]$ if and only if $r < a$. Symmetrically, the function h^+ is upper semicontinuous. Our preparatory steps on the lower and upper integrals then allow us to make sense of the following definition. The fact that $\int_{x\in X}^{-} h^-(x)d\mu \leq \int_{x\in X}^{+} h^+(x)d\mu$ is by Lemma 7.5, item 4.

Definition 7.6 For every τ-smooth measure μ on a topological space X, we define $\widetilde{\mu} \colon \mathcal{L}^{\mathbf{I}\mathbb{R}_+^{\star}} X \to \mathbf{I}\mathbb{R}_+^{\star}$ by $\widetilde{\mu}(h) \overset{\text{def}}{=} [\int_{x\in X}^{-} h^-(x)d\mu, \int_{x\in X}^{+} h^+(x)d\mu]$.

Proposition 7.7 *For every non-zero, bounded τ-smooth measure μ on a coherent topological space X, $\widetilde{\mu}$ is a continuous $\mathbf{I}\mathbb{R}_+^{\star}$-valuation.*

Proof. First, $\widetilde{\mu}$ is linear by Lemma 7.1 (items 1 and 2) and Lemma 7.5 (items 1 and 2). We verify that It it Scott-continuous. Let $(h_i)_{i\in I}$ be a directed family in $\mathcal{L}^{\mathbf{I}\mathbb{R}_+^{\star}} X$, with supremum h. We aim to show that $\widetilde{\mu}(h) = \sup_{i\in I}^{\uparrow} \widetilde{\mu}(h_i)$. On the one hand, $h^- = \sup_{i\in I}^{\uparrow} h_i^-$, so $\int_{x\in X}^{-} h^-(x)d\mu = \sup_{i\in I}^{\uparrow} \int_{x\in X}^{-} h_i^-(x)d\mu$ by Lemma 7.1, item 4. On the other hand, $h^+ = \inf_{i\in I}^{\downarrow} h_i^+$, so $\int_{x\in X}^{+} h^+(x)d\mu = \inf_{i\in I}^{\downarrow} \int_{x\in X}^{+} h_i^+(x)d\mu$ by Lemma 7.5, item 3. $\qquad\square$

Remark 7.8 We think of $\widetilde{\mu}$ as being really the measure μ, seen as a continuous $\mathbf{I}\mathbb{R}_+^{\star}$-valuation. Note in particular that for every bounded continuous map $h \colon X \to \mathbb{R}_+$, $\widetilde{\mu}([h, h]) = [\int_{x\in X} h(x)d\mu, \int_{x\in X} h(x)d\mu]$.

8 Continuous R-valuations and measures III: continuous $\mathbb{IR}_+^\star$-valuations as approximations of measures

Let us say that $[a, b]$ *approximates* x if and only if $a \leq x \leq b$, and that a continuous map $h \in \mathcal{L}^{\mathbb{IR}_+^\star} X$ *approximates* a measurable map $f \colon X \to \overline{\mathbb{R}}_+$ if and only if $h(x)$ approximates $f(x)$ for every $x \in X$.

We will say that a continuous $\mathbb{IR}_+^\star$-valuation ν *approximates* a measure μ on X if and only if, for every measurable map $f \colon X \to \overline{\mathbb{R}}_+$ and for every $h \in \mathcal{L}^{\mathbb{IR}_+^\star} X$ that approximates f, $\nu(h)$ approximates $\int_{x \in X}^- f(x) d\mu$.

Lemma 8.1 *For every non-zero, bounded τ-smooth measure μ on a coherent topological space X, $\widetilde{\mu}$ approximates μ.*

Proof. We consider any measurable map $f \colon X \to \overline{\mathbb{R}}_+$ and any $h \in \mathcal{L}^{\mathbb{IR}_+^\star} X$ that approximates f. We write $h(x)$ as $[h^-(x), h^+(x)]$ for every $x \in X$, so that $h^- \leq f \leq h^+$. Then $\int_{x \in X}^- h^-(x) d\mu \leq \int_{x \in X}^- f(x) d\mu \leq \int_{x \in X}^+ h^+(x) d\mu$, where the last inequality is by Lemma 7.5, item 4 applied to $f \leq h^+$. $\square$

Our objective is now to show that $\widetilde{\mu}$ is the most precise, namely the largest, continuous $\mathbb{IR}_+^\star$-valuation that approximates μ, under some reasonable assumptions. This will notably hold when the ambient space X is compact Hausdorff and second-countable, for example $[0, 1]$ with its usual, metric topology. More generally, this will hold when X is stably compact, second-countable, and contains a sufficiently nice support K of μ.

The value of restricting to second-countable spaces is the following.

Lemma 8.2 *Let X be a topological space, and $\mathcal{B}$ be a base of its topology that is closed under finite unions.*

(i) *For every compact saturated subset Q of X, and every open neighborhood U of Q, there is a $V \in \mathcal{B}$ such that $Q \subseteq V \subseteq U$.*

(ii) *Every compact saturated subset of X is equal to the intersection of the sets in $\mathcal{B}$ that contain it.*

In particular, if X is second-countable, then every compact saturated subset of X is measurable.

Proof. (i) We write U as the union of the sets $V \in \mathcal{B}$ that are included in V. This forms an open cover of Q, from which we can extract a finite subcover. Since $\mathcal{B}$ is closed under finite unions, there is a $V \in \mathcal{B}$ that contains Q and is included in U.

(ii) Let Q be compact saturated in X. Since Q is saturated, Q is the intersection of its open neighborhoods U. Then claim 2 follows from 1.

When $\mathcal{B}$ is countable, Q is then a countable intersection of open sets, so Q is measurable. $\square$

A space X is *stably compact* if and only if it is sober, locally compact, compact, and coherent. We let X^{patch} denote X with its *patch topology*, which is the smallest topology that contains the original open subsets of X and the complements of compact saturated subsets of X. When X is stably compact, and $\leq$ is its specialization ordering, $(X^{\mathsf{patch}}, \leq)$ is a *compact pospace*, meaning that X^{patch} is compact Hausdorff, and that the graph of $\leq$ is closed in $X^{\mathsf{patch}} \times X^{\mathsf{patch}}$. We say that a subset of X is *patch-open* if it is open in X^{patch}. Similarly, we use the terms *patch-closed, patch-compact*. If X is stably compact, then patch-closed and patch-compact are synonymous. We should add that the original open subsets of X can be recovered as those patch-open subsets that are upwards-closed with respect to $\leq$.

Example 8.3 $\mathbb{IR}_+^\star$ is stably compact in its Scott topology. Indeed, it is a continuous dcpo in which any pair of elements $[a, b]$ and $[c, d]$ with an upper bound (namely, such that $[a, b] \cap [c, d] \neq \emptyset$, or equivalently $\max(a, c) \leq \min(b, d)$) has a least upper bound (which is $[\max(a, c), \min(b, d)]$). That kind of continuous dcpo is called a *bc-domain*, and every bc-domain is stably compact [12, Fact 9.1.6].

Reasoning similarly, the larger dcpo $\mathbb{IR}^\star$ of all closed intervals in $\mathbb{R} \cup \{-\infty, \infty\}$, ordered by reverse inclusion, is also a bc-domain, hence is also stably compact. (To make it clear, note that $\mathbb{IR}^\star$ not only contains the usual intervals $[a, b]$ with $a, b \in \mathbb{R}$, $a \leq b$, but also $[-\infty, b]$, $[a, \infty]$ with $a, b \in \mathbb{R}$; finally, it has a least element $[-\infty, \infty]$.)

Both bc-domains are second-countable as well. Indeed, as continuous dcpos, they have a basis $\mathcal{B}$ of intervals with rational endpoints, and then the set of Scott-open sets $\Uparrow b$, $b \in \mathcal{B}$, forms a countable base of the Scott topology.

Patch-compact subsets K of stably compact subspaces X enjoy many nice properties. For example, their downward closure $\downarrow K$ in X is closed [12, Exercise 9.1.43]. In fact, we have the following, where K is *order-convex* if and only if for all x, y, z such that $y \leq x \leq z$, if $y, z \in K$ then $x \in K$.

Lemma 8.4 *For every compact, order-convex subset K of a stably compact space X, K is patch-compact if and only if $\downarrow K$ is closed. In that case, $\downarrow K$ is the closure $cl(K)$ of K in X, and $K = \uparrow K \cap \downarrow K$.*

Proof. We only have to show that if K is compact and order-convex and if $\downarrow K$ is closed, then it is patch-compact. This will be a consequence of the last equality $K = \uparrow K \cap \downarrow K$, since $\uparrow K$ is compact saturated, and we have assumed that $\downarrow K$ is closed, hence both are patch-closed; then K is patch-closed, too, hence patch-compact.

The inclusion $K \subseteq \uparrow K \cap \downarrow K$ is clear. Conversely, every $x \in \uparrow K \cap \downarrow K$ is such that $y \leq x \leq z$ for some $y, z \in K$, so order-convexity implies $y = z$, and therefore also $x = y = z$. In particular, x is in K. □

We will say that a subset K of a space X is Hausdorff if and only if it is Hausdorff as a subspace, namely with the subspace topology inherited from X. Since the specialization ordering of a Hausdorff space is equality, every Hausdorff subset is trivially order-convex.

Example 8.5 Let $X \overset{\text{def}}{=} \mathbb{IR}^\star$. Then the unit interval $K \overset{\text{def}}{=} [0, 1]$ embeds into X, provided that we equate every point $x \in [0, 1]$ with the interval $[x, x]$ in $\mathbb{IR}^\star$. It is Hausdorff, hence order-convex. Its downward closure $\downarrow K$ in X is the set of all intervals $[a, b]$ such that $[a, b] \cap [0, 1] \neq \emptyset$, or equivalently such that $\max(a, 0) \leq \min(b, 1)$, or equivalently $a \leq 1$ and $b \geq 0$. Then $\downarrow K$ is closed: for every directed family $([a_i, b_i])_{i \in I}$ in $\downarrow K$, its supremum $[a, b]$ is such that $a = \sup^\uparrow_{i \in I} a_i \leq 1$ and $b = \inf^\downarrow_{i \in I} b_i \geq 0$. By Lemma 8.4, $[0, 1]$ is patch-compact in $\mathbb{IR}^\star$.

When K is patch-compact in a stably compact space X, we have the serendipitous property that K, with the subspace topology, is stably compact, and that the patch topology on K is the subspace topology inherited from X^{patch} [12, Proposition 9.3.4]; also, the specialization ordering on K is the restriction of that on X.

The previous remark, together with the fact that $K^{\mathsf{patch}} = K$ if K is compact and Hausdorff (since every compact set is already closed in K), entails the following.

Lemma 8.6 *Let X be a stably compact space, and K be a Hausdorff, patch-compact subset of X. Then K has both the subspace topology inherited from X, and the subspace topology inherited from X^{patch}.* □

We also note that every Hausdorff subset is order-convex, since the specialization ordering on any Hausdorff space is equality.

Proposition 8.7 *Let K be a Hausdorff, patch-compact subset of a stably compact, second-countable space X. Let μ be a non-zero measure on X supported on K, and ν be a continuous $\mathbb{IR}^\star_+$-valuation on X. If ν approximates μ, then $\nu \leq \widetilde{\mu}$.*

Proof. We first note that, since X is second-countable, every measure on X is τ-smooth, in particular μ.

Let h be an arbitrary continuous map in $\mathcal{L}^{\mathbb{IR}^\star_+} X$, let us write $h(x)$ as $[h^-(x), h^+(x)]$ for every $x \in X$ (for short, $h = [h^-, h^+]$), and $\nu(h)$ as $[\nu^-(h), \nu^+(h)]$. We must show that $\nu^-(h) \leq \int^-_{x \in X} h^-(x) d\mu$, and that $\int^+_{x \in X} h^+(x) d\mu \leq \nu^+(h)$.

For the first claim, we note that $h(x)$ is the supremum of the chain of maps $[\min(h^-, r), h^+]$, $r \in \mathbb{R}_+$. For each $r \in \mathbb{R}_+$, $[\min(h^-, r), h^+]$ approximates the bounded lower semicontinuous (hence measurable) map $\min(h^-, r)$. Since it is bounded, $\int^-_{x \in X} \min(h^-(x), r) d\mu = \int_{x \in X} \min(h^-(x), r) d\mu$. By assumption, $\nu([\min(h^-, r), h^+])$ approximates $\int_{x \in X} \min(h^-(x), r) d\mu$. In particular, $\nu^-([\min(h^-, r), h^+]) \leq \int_{x \in X} \min(h^-(x), r) d\mu$. By taking suprema as r grows to infinity, and using the Scott-continuity of ν, hence of ν^-, $\nu^-(h) \leq \int^-_{x \in X} h^-(x) d\mu$.

For the second claim, we distinguish two cases. If h^+ is μ-bounded, then since h approximates h^+, $\nu(h)$ approximates $\int_{x\in X}^- h^+(x)d\mu$, which is equal to $\int_{x\in X}^+ h^+(x)d\mu$ by Lemma 7.5, item 5. In particular, $\int_{x\in X}^+ h^+(x)d\mu \leq \nu^+(h)$.

The only case where we have to work a bit is the final case, when h^+ is μ-unbounded. We let $Q \stackrel{\text{def}}{=} \uparrow K$. This is a compact saturated subset of X, and since X is second-countable, Q is measurable by Lemma 8.2. Moreover, Q is a support of μ, since Q contains K, which is already a support of μ.

$Q \cap \text{supp } \mu$ is then a measurable, compact support of μ. We claim that $\text{supp } \mu$ is included in the closure $cl(K)$ of K in X. Equivalently, we claim that every open set U that intersects $\text{supp } \mu$ also intersects K. Since U intersects $\text{supp } \mu$, by definition of $\text{supp } \mu$, we have $\mu(U) > 0$. If it did not intersect K, then U and the empty set would have the same intersection with K, and that would imply $\mu(U) = \mu(\emptyset) = 0$, which is impossible.

Since $\text{supp } \mu \subseteq cl(K)$, we obtain that $Q \cap \text{supp } \mu$ is included in $\uparrow K \cap cl(K)$. But $cl(K) = \downarrow K$ and $K = \uparrow K \cap \downarrow K$ by Lemma 8.4. (Recall that K is order-convex since Hausdorff.) Therefore, $Q \cap \text{supp } \mu$ is included in K. By Corollary 7.4, there is a point x_0 in $Q \cap \text{supp } \mu$, hence in K, such that $h^+(x_0) = \infty$.

Since K is compact Hausdorff hence locally compact, x_0 has a base of compact neighborhoods $(K_i)_{i\in I}$ in K. Each K_i is compact hence closed in K (since K is Hausdorff). Let C_i be the closure of K_i in X. Then $C_i \cap K = K_i$: the inclusion $K_i \subseteq C_i \cap K$ is clear; conversely, since K_i is closed in K, we can write it as $C \cap K$ for some closed subset C of X, and then $C \supseteq C_i$, so $K_i = C \cap K \supseteq C_i \cap K$.

The family $(K_i)_{i\in I}$ is filtered: for all $i, j \in I$, $K_i \cap K_j$ is a compact neighborhood of x_0, using the fact that the intersection of two compact sets in a Hausdorff space is compact; hence $K_i \cap K_j$ contains some K_k, $k \in I$. It follows that $(C_i)_{i\in I}$ is a filtered family of closed subsets of X.

Let $C \stackrel{\text{def}}{=} \bigcap^{\downarrow}_{i\in I} C_i$. This is a closed subset of X containing x_0. We claim that C is exactly the downward closure of x_0 in X. It only remains to show that $C \subseteq \downarrow x_0$. Let us assume the contrary: for some $x \in \bigcap^{\downarrow}_{i\in I} \downarrow K_i$, $x \not\leq x_0$. Then x_0 is in $(X \setminus \uparrow x) \cap K$, which is open in K. Indeed, $\uparrow x$ is compact saturated hence patch-closed in X, so $X \setminus \uparrow x$ is open in X^{patch}, and therefore $(X \setminus \uparrow x) \cap K$ is open in K, using Lemma 8.6. Since $(K_i)_{i\in I}$ is a base of neighborhoods of x_0 in K, some K_i is included in $(X \setminus \uparrow x) \cap K$. This is impossible, since $x \in \downarrow K_i$.

For every $i \in I$, let h_i be the function that maps every $x \in C_i$ to ∞, and all other points to $h^+(x)$. This is an upper semicontinuous map, since $h_i^{-1}([0,r[) = h^{+^{-1}}([0,r[) \setminus C_i$ for every $r \in \mathbb{R}_+$. The family $(h_i)_{i\in I}$ is filtered, since $(C_i)_{i\in I}$ is a filtered family of sets. Moreover, for every $x \in X$, $\inf^{\downarrow}_{i\in I} h_i(x) = h^+(x)$. If $x \leq x_0$, we argue as follows. First, $x \in \downarrow x_0 = C = \bigcap^{\downarrow}_{i\in I} C_i$, so that $\inf^{\downarrow}_{i\in I} h_i(x) = \inf^{\downarrow}_{i\in I} \infty = \infty$, while $h^+(x) \geq h^+(x_0) = \infty$, since upper semicontinuous maps are antitonic. If $x \not\leq x_0$, then x is not in $C = \bigcap^{\downarrow}_{i\in I} C_i$, so x is not in C_i for some $i \in I$, and therefore $h_i(x) = h^+(x)$. This implies that $\inf^{\downarrow}_{i\in I} h_i(x) \leq h^+(x)$, while the reverse inequality is obvious.

For every $i \in I$, $[h^-, h_i]$ approximates h_i, so $\nu([h^-, h_i])$ approximates $\int_{x\in X}^- h_i(x)d\mu$. In particular, $\int_{x\in X}^- h_i(x)d\mu \leq \nu^+([h^-, h_i])$. However, we claim that the left-hand side is equal to ∞, so that $\nu^+([h^-, h_i]) = \infty$. Indeed, $h_i(x)$ is equal to ∞ on C_i, hence on $K_i \subseteq C_i$, hence on the even smaller set $U_i \cap K$, so $\int_{x\in X}^- h_i(x)d\mu \geq \infty.\mu(U_i \cap K)$. (The latter makes sense because $K = \uparrow K \cap \downarrow K$ by Lemma 8.4, $\uparrow K$ is compact saturated hence measurable by Lemma 8.2, U_i is open and $\downarrow K$ are closed, hence are measurable.) Since K is a support of μ, $\mu(U_i \cap K) = \mu(U_i)$. We have an open set U_i that intersects $\text{supp } \mu$ (at x_0): by definition of the support, $\mu(U_i) > 0$. (Namely, if we had $\mu(U_i) = 0$, then U_i would be included in the largest open subset with zero μ-measure, which is the complement of $\text{supp } \mu$ by definition.) It follows that $\int_{x\in X}^- h_i(x)d\mu \geq \infty.\mu(K_i) \geq \infty.\mu(U_i) = \infty$.

Hence we have shown that $\nu^+([h^-, h_i]) = \infty$ for every $i \in I$. Taking suprema, and recalling that $h^+ = \inf^{\downarrow}_{i\in I} h_i$, hence that $\sup^{\uparrow}_{i\in I}[h^-, h_i] = [h^-, h^+] = h$, we obtain that $\nu^+(h) = \infty$. The inequality $\int_{x\in X}^+ h^+(x)d\mu \leq \nu^+(h)$ then follows trivially.   □

Theorem 8.8 *Let K be a Hausdorff, patch-compact subset of a stably compact, second-countable space*

X, and μ be a non-zero, bounded measure supported on *K*. Then μ is τ-smooth and $\widetilde{\mu}$ is the largest ("most precise") continuous $\mathbf{IR}_+^\star$-valuation that approximates μ.

Proof. First, μ is τ-smooth because *X* is second-countable. Second, $\widetilde{\mu}$ is a continuous $\mathbf{IR}_+^\star$-valuation by Proposition 7.7, it approximates μ by Lemma 8.1, and it is largest by Proposition 8.7.   □

9   The Lebesgue *R*-valuation on the unit interval

Let λ be Lebesgue measure on $[0, 1]$. By Theorem 8.8 with $X \overset{\text{def}}{=} K \overset{\text{def}}{=} [0, 1]$, $\widetilde{\lambda}$ is the most precise continuous $\mathbf{IR}_+^\star$-valuation that approximates λ. However, $\widetilde{\lambda}$ is not in $\mathbf{V}_m^{\mathbf{IR}_+^\star}([0, 1])$, by the following argument, whose details we leave to the reader. If $\widetilde{\lambda}$ were minimal, then its view from the left would be in $\mathbf{V}_m^{\overline{\mathbb{R}}_+}([0, 1])$, so λ would be a minimal valuation. Any minimal valuation is point-continuous, in the sense of Heckmann [15], because every simple valuation is point-continuous, and point-continuous valuations are closed under directed suprema. However, a valuation ν is point-continuous if and only if for every open set *U*, for every real number *r* such that $0 \leq r < \nu(U)$, there is a finite subset *A* of *U* such that $\nu(V)$ for every open neighborhood *V* of *A*; and λ fails to have this property, since every finite subset has open neighborhoods of arbitrarily small λ-measure.

Instead, we consider the image measure $j[\lambda]$ on $\mathbf{IR}^\star$, where *j* is the usual embedding of $[0, 1]$ inside $\mathbf{IR}^\star$, mapping *x* to the interval $[x, x]$. We will show that, contrarily to λ, $j[\lambda]$ *is* minimal.

This may sound somewhat paradoxical, considering that both have the same effect: drawing an interval at random with respect to measure $j[\lambda]$ means drawing an interval of the form $[x, x]$ with $x \in [0, 1]$ with probability 1, where *x* is drawn uniformly at random in $[0, 1]$, hence works just like λ; only the ambient space differs ($\mathbf{IR}^\star$ instead of $[0, 1]$).

We will say that $j[\lambda]$ is the *Lebesgue valuation on the unit interval* in $\mathbf{IR}^\star$. By Theorem 8.8 with $X \overset{\text{def}}{=} \mathbf{IR}^\star$ and $K \overset{\text{def}}{=} [0, 1]$, $\widetilde{j[\lambda]}$ is the continuous $\mathbf{IR}_+^\star$-valuation that approximates $j[\lambda]$ in the most precise possible way. It is a bounded, non-zero, and τ-smooth measure (because $\mathbf{IR}^\star$ is second-countable, see Example 8.3). The objective of this section is to show that $\widetilde{j[\lambda]}$ *is* in $\mathbf{V}_m^{\mathbf{IR}_+^\star}(\mathbf{IR}^\star)$.

To this end, we will show the stronger statement that $\widetilde{j[\lambda]}$ is the directed supremum of a countable chain of simple $\mathbf{IR}_+^\star$-valuations $\overline{\lambda}_n$, $n \in \mathbb{N}$.

We define $\overline{\lambda}_n$ on $\mathbf{IR}^\star$ as $\sum_{i=1}^{2^n} [\frac{1}{2^n}, \frac{1}{2^n}] \times \delta_{[\frac{i-1}{2^n}, \frac{i}{2^n}]}$—which we will write more simply as $\sum_{i=1}^{2^n} \frac{1}{2^n} \delta_{[\frac{i-1}{2^n}, \frac{i}{2^n}]}$, equating points $a \in \mathbb{R} \cup \{-\infty, \infty\}$ with intervals $[a, a]$.

Lemma 9.1 *The simple $\mathbf{IR}_+^\star$-valuations $\overline{\lambda}_n$, $n \in \mathbb{N}$, form an ascending chain in $\mathbf{V}_f^{\mathbf{IR}_+^\star}(\mathbf{IR}^\star)$.*

Proof. It suffices to show that $\overline{\lambda}_n \leq \overline{\lambda}_{n+1}$. For every $h \in \mathcal{L}^{\mathbf{IR}_+^\star}(\mathbf{IR}^\star)$,

$$
\begin{aligned}
\overline{\lambda}_n(h) &= \sum_{i=1}^{2^n} \frac{1}{2^n} \times h\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right) \\
&= \sum_{i=1}^{2^n} \left(\frac{1}{2^{n+1}} \times h\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right) + \frac{1}{2^{n+1}} \times h\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right)\right) \\
&\leq \sum_{i=1}^{2^n} \left(\frac{1}{2^{n+1}} \times h\left(\left[\frac{2i-2}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\right]\right) + \frac{1}{2^{n+1}} \times h\left(\left[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\right]\right)\right) \\
&= \overline{\lambda}_{n+1}(h).
\end{aligned}
$$

The second line is justified by the fact that $\times$ distributes over $+$. The inequality on the third line follows from the fact that $[\frac{i-1}{2^n}, \frac{i}{2^n}]$ is below (contains) both $[\frac{2i-2}{2^{n+1}}, \frac{2i-1}{2^{n+1}}]$ and $[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}]$, and that *h*, product and addition are monotonic. The last line follows by rearranging the sum.   □

The chain $(\overline{\lambda}_n)_{n\in\mathbb{N}}$ then has a supremum in $\mathbf{V}^{\mathbb{IR}^\star_+}(\mathbb{IR}^\star)$, which is in $\mathbf{V}_m^{\mathbb{IR}^\star_+}(\mathbb{IR}^\star)$ by definition of the latter, since every $\overline{\lambda}_n$ is simple.

Definition 9.2 Let $\overline{\lambda}$ be $\sup^\uparrow_{n\in\mathbb{N}}\overline{\lambda}_n$.

For every $h \in \mathcal{L}^{\mathbb{IR}^\star_+}(\mathbb{IR}^\star)$, we have:

$$\overline{\lambda}(h) = \sup^\uparrow_{n\in\mathbb{N}} \overline{\lambda}_n(h)$$
$$= \left[\sup^\uparrow_{n\in\mathbb{N}} \sum_{i=1}^{2^n} \frac{1}{2^n} h^- \left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right), \inf^\downarrow_{n\in\mathbb{N}} \sum_{i=1}^{2^n} \frac{1}{2^n} h^+ \left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right)\right]. \tag{6}$$

Theorem 9.3 *The continuous $\mathbb{IR}^\star_+$-valuation $\overline{\lambda}$ is the largest ("most precise") continuous $\mathbb{IR}^\star_+$-valuation $\widetilde{j[\lambda]}$ that approximates Lebesgue measure on the unit interval in $\mathbb{IR}^\star$.*

In particular, $\widetilde{j[\lambda]} = \overline{\lambda}$ is in $\mathbf{V}_m^{\mathbb{IR}^\star_+}(\mathbb{IR}^\star)$.

Proof. The interval $[0,1]$, with its usual ordering, is a continuous dcpo, and its way-below relation $\ll$ is such that $x \ll y$ if and only if $x = 0$ or $x < y$. For every $n \in \mathbb{N}$, for every $x \in [0,1]$, let $j_n^-(x)$ be the largest integer multiple of $\frac{1}{2^n}$ way-below x; explicitly, $j_n^-(x) \stackrel{\text{def}}{=} \frac{i}{2^n}$ if $x \in \left]\frac{i}{2^n}, \frac{i+1}{2^n}\right]$, $i \in \{1, 2, \cdots, 2^n - 1\}$, and $j_n^-(x) \stackrel{\text{def}}{=} 0$ if $x \in \left[0, \frac{1}{2^n}\right]$. By the definition of $\ll$, j_n^- is Scott-continuous from $[0,1]$ to $\mathbb{R}\cup\{-\infty,\infty\}$ (both taken with their usual orderings). This implies that j_n is lower semicontinuous from $[0,1]$ to $\mathbb{R}\cup\{-\infty,\infty\}$, with their usual, Hausdorff topologies.

Let also $j_n^+(x) \stackrel{\text{def}}{=} 1 - j_n^-(1-x)$. The function j_n^+ is an upper semicontinuous map, and $j_n^-(x) \leq x \leq j_n^+(x)$ for every $x \in X$. This implies that, if we define $j_n(x)$ as $[j_n^-(x), j_n^+(x)]$, j_n is a continuous map from $[0,1]$ to $\mathbb{IR}^\star$, and $j_n \leq j$.

One checks easily that $j_n^- \leq j_{n+1}^-$, hence $j_n^+ \geq j_{n+1}^+$, and therefore $j_n \leq j_{n+1}$. It follows that $(j_n)_{n\in\mathbb{N}}$ is an increasing chain of continuous maps. Moreover $\sup^\uparrow_{n\in\mathbb{N}} j_n = j$. Indeed, for every $x \in X$, $\sup^\uparrow_{n\in\mathbb{N}} j_n^-(x) = x$, and $\inf^\downarrow_{n\in\mathbb{N}} j_n^+(x) = 1 - \sup^\uparrow_{n\in\mathbb{N}} j_n^-(1-x) = 1 - (1-x) = x$.

Let now $h \stackrel{\text{def}}{=} [h^-, h^+]$ be any element of $\mathcal{L}^{\mathbb{IR}^\star_+}(\mathbb{IR}^\star)$. We wish to show that $\widetilde{j[\lambda]}(h) = \overline{\lambda}(h)$, namely that $\left[\int_{y\in\mathbb{IR}^\star}^- h^-(y)dj[\lambda], \int_{y\in\mathbb{IR}^\star}^+ h^+(y)dj[\lambda]\right] = \overline{\lambda}(h)$.

Note that h^- is lower semicontinuous, or equivalently Scott-continuous from $\mathbb{IR}^\star$ to $\mathbb{IR}^\star_+$. We have:

$$\int_{y\in\mathbb{IR}^\star}^- h^-(y)dj[\lambda] = \int_{x\in[0,1]}^- h^-(j(x))d\lambda \qquad \text{by the change of variable formula (4)}$$

$$= \int_{x\in[0,1]}^- \sup^\uparrow_{n\in\mathbb{N}} h^-(j_n(x))d\lambda \qquad \text{since } h^- \text{ is Scott continuous}$$

$$= \sup^\uparrow_{n\in\mathbb{N}} \int_{x\in[0,1]}^- h^-(j_n(x))d\lambda \qquad \text{by Lemma 7.1, item 3 (or 4).}$$

The function $h^- \circ j_n$ is piecewise constant: it takes the value $h^-(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right])$ on the open subinterval $\left]\frac{i-1}{2^n}, \frac{i}{2^n}\right[$, $1 \leq i \leq 2^n$. Since the values in $[0,1]$ that are not in one of those open subintervals, namely the integer multiples of $\frac{1}{2^n}$, form a set of Lebesgue measure 0, they do not contribute to the integral, so:

$$\int_{y\in\mathbb{IR}^\star}^- h^-(y)dj[\lambda] = \sup^\uparrow_{n\in\mathbb{N}} \sum_{i=1}^{2^n} \frac{1}{2^n} h^- \left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right),$$

and we recognize the left endpoint of the final interval of (6).

We now deal with the right endpoint. For that, we need to understand what the supports of $j[\lambda]$ are on $\mathbb{IR}^\star$. Let K be the image of $[0,1]$ by j in $\mathbb{IR}^\star$. We have seen in Example 8.5 that $\downarrow K$ is closed in $\mathbb{IR}^\star$, hence that K is patch-compact in $\mathbb{IR}^\star$, using Lemma 8.4.

For every open subset U of $\mathbb{IR}^\star$, $j[\lambda](U) > 0$ if and only if $\lambda(j^{-1}(U)) > 0$, if and only if $j^{-1}(U)$ is non-empty. Indeed, the Lebesgue measure of any non-empty open set is non-zero. Now $j^{-1}(U)$ is empty if and only if U does not intersect K, if and only if U does not intersect the closure of K, which is $\downarrow K$, since $\downarrow K$ is closed. This entails that $\mathrm{supp}\, j[\lambda] = \downarrow K$.

We note that K is a support of $j[\lambda]$. The easy argument is as follows. First, K is a set of maximal elements of $\mathbb{IR}^\star$, so $K = \uparrow K$; by Lemma 8.2, it is measurable. In order to show that K is a support of $j[\lambda]$, it then suffices to observe that $j[\lambda](K) = 1$, and this follows from the fact that $j[\lambda](K) = \lambda(j^{-1}(K)) = \lambda([0,1]) = 1$.

We claim that every compact saturated support Q of $j[\lambda]$ must contain K. We argue as follows. By Lemma 8.6, and since Q is patch-closed in X^{patch} (where $X \stackrel{\text{def}}{=} \mathbb{IR}^\star$), $Q \cap K$ is closed in K. If Q does not contain K, then $Q \cap K$ is a proper subset of K. Let $U \stackrel{\text{def}}{=} K \setminus Q$: this is a non-empty open subset of K, and therefore $j^{-1}(U)$ is a non-empty subset of $[0,1]$. Hence $\lambda(j^{-1}(U)) > 0$, so $j[\lambda](U) > 0$. It follows that $j[\lambda](Q \cap K) = 1 - j[\lambda](U) < 1$. Since K is a support of $j[\lambda]$, we obtain that $j[\lambda](Q) = j[\lambda](Q \cap K) < 1$, and that contradicts that Q is a support of $j[\lambda]$.

It follows that $\uparrow K \,(= K)$ is the smallest compact saturated support of $j[\lambda]$. In that case, and with $\mu \stackrel{\text{def}}{=} j[\lambda]$, the definition of μ-boundedness simplifies: h^+ is μ-bounded if and only if h^+ is bounded on $\uparrow K \cap \mathrm{supp}\, \mu = \uparrow K \cap \downarrow K = K$.

It is even easier to show that $h^+ \circ j$ is λ-bounded if and only if it is bounded on $[0,1]$. Indeed, $\mathrm{supp}\, \lambda = [0,1]$, and therefore the only compact (hence closed) support Q of λ is $[0,1]$, so that there is only one possible set $Q \cap \mathrm{supp}\, \lambda$ to be considered, namely $[0,1]$.

If h^+ is μ-unbounded, then $\int_{y \in \mathbb{IR}^\star} h^+(y) dj[\lambda] = \infty$ by definition. By Corollary 7.4 with $Q \stackrel{\text{def}}{=} \uparrow K = K$, there is a point $[x,x] \in Q \cap \mathrm{supp}\, \mu = K$ such that $h^+([x,x]) = \infty$. For every $n \in \mathbb{N}$, there is a natural number i such that $x \in [\frac{i-1}{2^n}, \frac{i}{2^n}]$, $1 \le i \le 2^n$. Then $[\frac{i-1}{2^n}, \frac{i}{2^n}] \le [x,x]$, and since every upper semicontinuous map is antitonic, $h^+([\frac{i-1}{2^n}, \frac{i}{2^n}]) \ge h^+([x,x]) = \infty$. It follows that $\sum_{i=1}^{2^n} \frac{1}{2^n} h^+([\frac{i-1}{2^n}, \frac{i}{2^n}]) = \infty$. Since that holds for every $n \in \mathbb{N}$, $\inf^\downarrow_{n \in \mathbb{N}} \sum_{i=1}^{2^n} \frac{1}{2^n} h^+([\frac{i-1}{2^n}, \frac{i}{2^n}])$, which is the right endpoint of the final interval of (6), is equal to ∞, hence to $\int_{y \in \mathbb{IR}^\star} h^+(y) dj[\lambda]$.

It remains to deal with the case where h^+ is μ-bounded, and we have seen that this means that h^+ is bounded on K. Let $r \in \mathbb{R}_+$ be such that for every $y \in K$, $h^+(y) < r$. Hence K is included in the open set $h^{+-1}([0,r[)$.

For every $n \in \mathbb{N}$, let $Q_n \stackrel{\text{def}}{=} \uparrow\{[\frac{i-1}{2^n}, \frac{i}{2^n}] \mid i \in \{1, 2, \cdots, 2^n\}\}$, a compact saturated set that contains K. (The upward closure of any finite set is compact saturated.) It is also easy to see that any point of $\bigcap^\downarrow_{n \in \mathbb{N}} Q_n$ is of the form $[x,x]$ with $x \in [0,1]$, so $\bigcap^\downarrow_{n \in \mathbb{N}} Q_n = K$. Since $\mathbb{IR}^\star$ is sober, it is well-filtered, and therefore $K \subseteq h^{+-1}([0,r[)$ implies the existence of an $n \in \mathbb{N}$ such that $Q_n \subseteq h^{+-1}([0,r[)$. (Here is an alternate argument that avoids well-filteredness. Q_n is compact saturated hence closed in $\mathbb{IR}^{\star\mathsf{patch}}$. The complements of the sets Q_n then form an open cover of the complement of $h^{+-1}([0,r[)$ in $\mathbb{IR}^\star$. That complement is closed hence compact in $\mathbb{IR}^{\star\mathsf{patch}}$. We can then extract a finite subcover, and since the sets Q_n form a chain, there is a single $n \in \mathbb{N}$ such that the complement of Q_n contains the complement of $h^{+-1}([0,r[)$.)

Let n_0 be the natural number n that we have just found. Then we perform the following computation:

$$\int_{y\in\mathbf{IR}^\star}^+ h^+(y)dj[\lambda] = \int_{y\in\mathbf{IR}^\star}^- h^+(y)dj[\lambda] \qquad\qquad \text{since } h^+ \text{ is } j[\lambda]\text{-bounded}$$

$$= \int_{x\in[0,1]}^- h^+(j(x))d\lambda \qquad\qquad \text{by the change of variable formula (4)}$$

$$= \int_{x\in[0,1]}^+ h^+(j(x))d\lambda.$$

The latter is justified by the fact that $h^+ \circ j$ is bounded (by r) on $[0,1]$, hence is λ-bounded, as we have seen above.

Let us proceed. The first step below is justified by the fact that h^+ is upper semicontinuous, hence Scott-continuous from $\mathbf{IR}^\star$ to $\overline{\mathbb{R}}_+$ with the opposite ordering; in particular, h^+ maps directed suprema to filtered infima:

$$\int_{x\in[0,1]}^+ h^+(j(x))d\lambda = \int_{x\in[0,1]}^+ \inf_{n\in\mathbb{N}}^\downarrow h^+(j_n(x))d\lambda$$

$$= \inf_{n\in\mathbb{N}}^\downarrow \int_{x\in[0,1]}^+ h^+(j_n(x))d\lambda \qquad\qquad \text{by Lemma 7.5, item 3}$$

$$= \inf_{n\in\mathbb{N}, n>n_0}^\downarrow \int_{x\in[0,1]}^+ h^+(j_n(x))d\lambda,$$

where we have restricted the infimum to the indices above n_0 in the last line. (The infimum of a chain coincides with the infimum of any coinitial chain.)

For every $n > n_0$, for every $x \in [0,1]$, $j_n(x)$ is an interval of the form $[\frac{i-1}{2^n}, \frac{i}{2^n}]$ (if x is in the interval $]\frac{i-1}{2^n}, \frac{i}{2^n}[$, or if $x = 0$, or if $x = 1$), or of the form $[\frac{i-1}{2^n}, \frac{i+1}{2^n}]$ (if x is exactly $\frac{i}{2^n}$, $1 \le i \le 2^n - 1$). Since $n > n_0$, whichever the case is, $j_n(x)$ is in Q_{n_0}, hence in $h^{+-1}([0,r[)$. This means that $h^+ \circ j_n$ is a bounded function, and therefore that $\int_{x\in[0,1]}^+ h^+(j_n(x))d\lambda$ is the ordinary Lebesgue integral $\int_{x\in[0,1]} h^+(j_n(x))d\lambda$. Since $h^+ \circ j_n$ is piecewise constant (as with $h^- \circ j_n$, earlier on), that Lebesgue integral is equal to $\sum_{i=1}^{2^n} \frac{1}{2^n} h^+([\frac{i-1}{2^n}, \frac{i}{2^n}])$. Therefore:

$$\int_{y\in\mathbf{IR}^\star}^+ h^+(y)dj[\lambda] = \inf_{n\in\mathbb{N}, n>n_0}^\downarrow \sum_{i=1}^{2^n} \frac{1}{2^n} h^+\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right)$$

$$= \inf_{n\in\mathbb{N}}^\downarrow \sum_{i=1}^{2^n} \frac{1}{2^n} h^+\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]\right),$$

and we recognize the right endpoint of the final interval of (6). $\qquad\qquad\square$

10   Conclusion

We have proposed an extension of the notion of continuous valuation, or measure, with values in suitable domains beyond $\overline{\mathbb{R}}_+$. We have argued that continuous R-valuations, where R is a so-called Abelian d-rag, provide such an extension. Beyond $\overline{\mathbb{R}}_+$, a particularly interesting Abelian d-rag is the domain of intervals $\mathbf{IR}^\star_+$, and we have shown that there is an ample supply of continuous $\mathbf{IR}^\star_+$-valuations stemming from measures.

There are many pending questions. For example, is $\mathbf{V}^R(X)$ a *continuous* dcpo, provided that X is a continuous dcpo and R is a continuous Abelian d-rag? Is there a form of the Fubini theorem for continuous R-valuations, beyond the one we have obtained for minimal R-valuations? None of the usual proof arguments, in realms of measures or of continuous valuations, seems to apply.

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