

# On $k$ -ranks of Topological Spaces

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## Abstract

In this paper, the concepts of  $K$ -subset systems and  $k$ -well-filtered spaces are introduced, which provide another uniform approach to  $d$ -spaces,  $s$ -well-filtered spaces (i.e.,  $\mathcal{U}_S$ -admissibility) and well-filtered spaces. We prove that the  $k$ -well-filtered reflection of any  $T_0$  space exists. Meanwhile, we propose the definition of  $k$ -rank, which is an ordinal that measures how many steps from a  $T_0$  space to a  $k$ -well-filtered space. Moreover, we derive that for any ordinal  $\alpha$ , there exists a  $T_0$  space whose  $k$ -rank equals to  $\alpha$ . One immediate corollary is that for any ordinal  $\alpha$ , there exists a  $T_0$  space whose  $d$ -rank (respectively,  $wf$ -rank) equals to  $\alpha$ .

*Keywords:*  $k$ -well-filtered space,  $k$ -Rudin set,  $k$ -well-filtered reflection,  $k$ -rank

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## 1 Introduction

In non-Hausdorff topological spaces and domain theory,  $d$ -spaces and well-filtered spaces are two important classes of spaces. Let  $\mathbf{Top}_0$  be the category of all  $T_0$  spaces,  $\mathbf{Top}_d$  the category of all  $d$ -spaces and  $\mathbf{Top}_w$  the category of all well-filtered spaces. It is well-known that  $\mathbf{Top}_d$  and  $\mathbf{Top}_w$  are reflective in  $\mathbf{Top}_0$ , respectively. Different ways for constructing  $d$ -completions and well-filtered reflections of  $T_0$  spaces were found in [3,10,12,17]. In [3], Ershov introduced one way to get  $d$ -completions of  $T_0$  spaces using the equivalent classes of directed subsets, he called it  $d$ -rank which is an ordinal that measures how many steps from a  $T_0$  space to a  $d$ -space. Inspired by his method, in [10], Liu, Li and Wu proposed one way to get well-filtered reflections of  $T_0$  spaces using the equivalent classes of Rudin subsets, they called it  $wf$ -rank, which is an ordinal that measures how far a  $T_0$  space is from being a well-filtered space.

In [16], based on irreducible subset systems, Xu provided a uniform approach to  $d$ -spaces, sober spaces and well-filtered spaces, and developed a general framework for dealing with all these spaces. In this paper, we will provide another uniform approach to  $d$ -spaces and well-filtered spaces and develop a general framework for dealing with all these spaces. Similar to the concept of irreducible subset systems in [16], we propose the concepts of  $K$ -subset systems and  $k$ -well-filtered spaces. For a  $K$ -subset system  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  and a  $T_0$  space  $X$ ,  $X$  is called  $k$ -well-filtered if for any open set  $U$  and a filtered family  $\mathcal{K} \subseteq Q_k(X)$ ,  $\bigcap \mathcal{K} \subseteq U$  implies  $K \subseteq U$  for some  $K \in \mathcal{K}$ . The category of all  $k$ -well-filtered spaces with continuous

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mappings is denoted by  $\mathbf{Top}_k$ . It is not difficult to verify that  $d$ -spaces and well-filtered spaces are two special kinds of  $k$ -well-filtered spaces. Moreover, we find that  $s$ -well-filtered spaces (i.e.,  $\mathcal{U}_S$ -admissibility in [6]) is also a kind of  $k$ -well-filtered spaces which is different from  $d$ -spaces and well-filtered spaces. Just like directed subsets and Rudin subsets, the concept of  $k$ -Rudin sets will be introduced. Moreover, for a  $K$ -subset system  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$ , we use the equivalent classes of  $k$ -Rudin sets to construct the  $k$ -well-filtered reflections of  $T_0$  spaces. Meanwhile, we introduce the concept of  $k$ -rank, which is an ordinal that measures how far a  $T_0$  space can become a  $k$ -well-filtered space. For a  $T_0$  space  $X$ , we get that there exists an ordinal  $\alpha$  such that the  $k$ -rank of  $X$  is equal to  $\alpha$ .

In [3] and [8], for any ordinal  $\alpha$ , there exists a  $T_0$  space whose  $d$ -rank (respectively,  $wf$ -rank) equals to  $\alpha$ . Consider a  $T_0$  space whose  $k$ -rank equals to  $\alpha$  may be more complex, because we know little about  $Q_k(X)$ . We have to find suitable conditions to characterize a class of  $T_0$  spaces whose  $k$ -rank equals to  $\alpha$ . It turns out that finding these  $T_0$  spaces is the hard part of our task, but how to prove the results is relatively simple.

Finally, we obtain that for any ordinal  $\alpha$ , there exists a  $T_0$  space whose  $k$ -rank equals to  $\alpha$ .

## 2 Preliminaries

First, we briefly recall some standard definitions and notations to be used in this paper, for further details see [1], [4], [5] and [7].

Let  $P$  be a poset and  $A \subseteq P$ . We denote  $\uparrow A = \{x \in P \mid x \geq a \text{ for some } a \in A\}$  and  $\downarrow A = \{x \in P \mid x \leq a \text{ for some } a \in A\}$ . For every  $a \in P$ , we denote  $\uparrow\{a\} = \uparrow a = \{x \in P \mid x \geq a\}$  and  $\downarrow\{a\} = \downarrow a = \{x \in P \mid x \leq a\}$ .  $A$  is called an *upper set* (resp., a *lower set*) if  $A = \uparrow A$  (resp.,  $A = \downarrow A$ ).  $A$  is called *directed* provided that it is nonempty and every finite subset of  $A$  has an upper bound in  $A$ . The set of all directed sets of  $P$  is denoted by  $\mathcal{D}(P)$ . Moreover, the set of all nonempty finite sets in  $P$  is denoted by  $P^{<\omega}$ .

A poset  $P$  is called a *dcpo* if every directed subset  $D$  in  $P$  has a supremum. A subset  $U$  of  $P$  is called *Scott open* if (1)  $U = \uparrow U$  and (2) for any directed subset  $D$  for which  $\vee D$  exists,  $\vee D \in U$  implies  $D \cap U \neq \emptyset$ . All Scott open subsets of  $P$  form a topology, we call it the *Scott topology* on  $P$  and denoted by  $\sigma(P)$ .

For a  $T_0$  space  $X$ , let  $\mathcal{O}(X)$  (resp.,  $\Gamma(X)$ ) be the set of all open subsets (resp., closed subsets) of  $X$ . For a subset  $A$  of  $X$ , the closure of  $A$  is denoted by  $\text{cl}(A)$  or  $\overline{A}$ . We use  $\leq_X$  to represent the specialization order of  $X$ , that is,  $x \leq_X y$  iff  $x \in \overline{\{y\}}$ . A subset  $B$  of  $X$  is called *saturated* if  $B$  equals the intersection of all open sets containing it (equivalently,  $B$  is an upper set in the specialization order). Let  $S(X) = \{\{x\} \mid x \in X\}$ ,  $S_c(X) = \{\downarrow x \mid x \in X\}$  and  $\mathcal{D}_c(X) = \{\overline{D} \mid D \in \mathcal{D}(X)\}$ . A  $T_0$  space  $X$  is called a *d-space* (i.e., *monotone convergence space*) if  $X$  (with the specialization order) is a *dcpo* and  $\mathcal{O}(X) \subseteq \sigma(X)$  ([4]). The category of all  $d$ -spaces with continuous mappings is denoted by  $\mathbf{Top}_d$ .

For a  $T_0$  space  $X$ , let  $\mathcal{K}$  be a filtered family under the inclusion order in  $Q(X)$ , which is denoted by  $\mathcal{K} \subseteq_{\text{filt}} Q(X)$ , i.e., for any  $K_1, K_2 \in Q(X)$ , there exists  $K_3 \in Q(X)$  such that  $K_3 \subseteq K_1 \cap K_2$ .  $X$  is called *well-filtered* if for any open subset  $U$  and any  $\mathcal{K} \subseteq_{\text{filt}} Q(X)$ ,  $\bigcap \mathcal{K} \subseteq U$  implies  $K \subseteq U$  for some  $K \in \mathcal{K}$ . The category of all well-filtered spaces with continuous mappings is denoted by  $\mathbf{Top}_w$  ([14]).

In what follows,  $\mathbf{K}$  always refers to a full subcategory  $\mathbf{Top}_0$  that contains  $\mathbf{Sob}$ , the full subcategory of sober spaces. The objects of  $\mathbf{K}$  are called  *$\mathbf{K}$ -spaces*.

**Definition 2.1** [14] Let  $X$  be a  $T_0$  space. A  *$\mathbf{K}$ -reflection* of  $X$  is a pair  $\langle \widehat{X}, \mu \rangle$  comprising a  $\mathbf{K}$ -space  $\widehat{X}$  and a continuous mapping  $\mu: X \rightarrow \widehat{X}$  satisfying that for any continuous mapping  $f: X \rightarrow Y$  to a  $\mathbf{K}$ -space, there exists a unique continuous mapping  $f^*: \widehat{X} \rightarrow Y$  such that  $f^* \circ \mu = f$ , that is, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \widehat{X} \\ & \searrow f & \downarrow f^* \\ & & Y \end{array}$$

By a standard argument,  $\mathbf{K}$ -reflections, if they exist, are unique up to homeomorphism. We shall use  $X^k$  to denote the space of the  $\mathbf{K}$ -reflection of  $X$  if it exists.

For  $\mathbf{K} = \mathbf{Top}_w$ , the  $\mathbf{K}$ -reflection of  $X$  is called the *well-filterification* or *well-filtered reflection* of  $X$ , we denote it by  $H_{wf}(X)$  if the well-filterification of  $X$  exists. For  $\mathbf{K} = \mathbf{Top}_d$ , the  $\mathbf{K}$ -reflection of  $X$  is called the *d-completion* of  $X$ , we denote it by  $H_d(X)$  if the *d-completion* of  $X$  exists.

**Definition 2.2** [13] Let  $X = (X, \tau)$  be a topological space and  $A \subseteq X$ .  $A$  is called strongly compact in  $X$  if for each  $U \in \tau$  with  $A \subseteq U$ , there is  $F \in X^{<\omega}$  such that  $A \subseteq \uparrow_\tau F \subseteq U$ .

**Proposition 2.3** [6] *Every finite set is strongly compact, and every strongly compact set is compact.*

**Proposition 2.4** [6] *A is strongly compact if and only if  $\uparrow A$  is so.*

We use  $Q_s(X)$  to denote the set of all nonempty strongly compact saturated subsets of  $X$ .  $X$  is called *s-well-filtered* (i.e.,  $U_S$ -admissibility in [6]) if it is  $T_0$ , and for any open subset  $U$  and  $\mathcal{K} \subseteq_{filt} Q_s(X)$ ,  $\bigcap \mathcal{K} \subseteq U$  implies  $K \subseteq U$  for some  $K \in \mathcal{K}$ . The category of all *s-well-filtered* spaces with continuous mappings is denoted by  $\mathbf{Top}_{s-w}$ .

### 3 *k*-well-filtered spaces

In this section, we provide a uniform approach to *d*-spaces and well-filtered spaces and develop a general framework for dealing with all these spaces.

**Definition 3.1**  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  is called a *C-subset system* if  $S^u(X) \subseteq Q_k(X) \subseteq Q(X)$  for all  $X \in ob(\mathbf{Top}_0)$ , where  $S^u(X) = \{\uparrow x \mid x \in X\}$ .

**Definition 3.2** Let  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  be a *C-subset system* and  $X$  a  $T_0$  space. A nonempty subset  $A$  is said to have *k-Rudin property*, if there exists  $\mathcal{K} \subseteq_{filt} Q_k(X)$  such that  $A$  is a minimal closed set that intersects all members of  $\mathcal{K}$ . We call such a set *k-Rudin* or *k-Rudin set*. Let  $K^R(X) = \{A \subseteq X \mid A \text{ has } k\text{-Rudin property}\}$  and  $K_c^R(X) = K^R(X) \cap \Gamma(X)$ .

For  $Q_k(X) = Q(X)$ , a *k-Rudin set* of  $X$  is called a *Rudin set* (i.e., *KF set*) of  $X$ . The set of all Rudin sets of  $X$  is denoted by  $\overline{KF}(X)$ .  $RD(X) = \overline{KF}(X) \cap \Gamma(X)$ .

**Proposition 3.3** Let  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  be a *C-subset system* and  $X$  a  $T_0$  space. Then  $\mathcal{D}(X) \subseteq K^R(X) \subseteq \overline{KF}(X)$ .

**Proof.** Clearly,  $K^R(X) \subseteq \overline{KF}(X)$ . Now we prove that every directed subset  $D$  of  $X$  is a *k-Rudin set*. Let  $\mathcal{K} = \{\uparrow d \mid d \in D\}$ . Then  $\mathcal{K} \subseteq Q_k(X)$  is filtered and  $\overline{D}$  intersects all members of  $\mathcal{K}$ . Assume that  $A$  is a closed subset in  $X$  and intersects all members of  $\mathcal{K}$ . This means that  $A \cap \uparrow d \neq \emptyset$  for all  $d \in D$ . Since  $A$  is closed, it is a lower set, then  $d \in A$  for all  $d \in D$ . Hence,  $\overline{D} \subseteq A$ . So  $\overline{D}$  is a minimal closed set that intersects all members of  $\mathcal{K}$ .  $\square$

**Definition 3.4** A *C-subset system*  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  is called a *K-subset system* provided that for any  $T_0$  spaces  $X, Y$  and any continuous mapping  $f : X \rightarrow Y$ ,  $f(A) \in K^R(Y)$  for all  $A \in K^R(X)$ .

**Definition 3.5** Let  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  be a *K-subset system* and  $X$  a  $T_0$  space.  $X$  is called *k-well-filtered* if for any open set  $U$  and  $\mathcal{K} \subseteq_{filt} Q_k(X)$ ,  $\bigcap \mathcal{K} \subseteq U$  implies  $K \subseteq U$  for some  $K \in \mathcal{K}$ . The category of all *k-well-filtered* spaces with continuous mappings is denoted by  $\mathbf{Top}_k$ .

In the following, we give some special *k-well-filtered* spaces and their relations with  $\mathbf{Top}_d$ ,  $\mathbf{Top}_w$  and  $\mathbf{Top}_{s-w}$ , respectively.

For a *C-subset system*  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  and a  $T_0$  space  $X$ , here are some important examples of  $Q_k(X)$ :

- (1)  $Q_k(X) = S^u(X)$  (i.e.,  $Q_k(X) = \{\uparrow x \mid x \in X\}$ ).
- (2)  $Q_k(X) = Q_f(X)$  (i.e.,  $Q_k(X) = \{\uparrow F \mid \emptyset \neq F \in X^{<\omega}\}$ ).

- (3)  $Q_k(X) = Q_s(X)$  (i.e.,  $Q_k(X) = \{A \mid A \text{ is a nonempty strongly compact saturated subset in } X\}$ ).
- (4)  $Q_k(X) = Q(X)$  (i.e.,  $Q_k(X)$  is the set of all nonempty compact saturated subsets in  $X$ ).

Let  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  be a  $C$ -subset system. For any  $T_0$  space  $X$ , if  $Q_k(X) = S^u(X)$  (i.e.,  $Q_k(X) = Q_f(X)$ ), then it follows directly from Definition 1, Example 1(1) and Theorem 1 in [9] that  $X$  is  $k$ -well-filtered iff  $X$  is a  $d$ -space. If  $Q_k(X) = Q(X)$ , it is trivial that  $X$  is  $k$ -well-filtered iff  $X$  is well-filtered. In the case  $Q_k(X) = Q_s(X)$ ,  $k$ -well-filtered spaces are exactly  $s$ -well-filtered spaces.

From the above, for a  $K$ -subset system  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$ , it is not difficult to see that well-filtered spaces are  $k$ -well-filtered spaces and  $k$ -well-filtered spaces are  $d$ -spaces. That is

$$\mathbf{Top}_w \subseteq \mathbf{Top}_k \subseteq \mathbf{Top}_d.$$

In particular, well-filtered spaces are  $s$ -well-filtered spaces and  $s$ -well-filtered spaces are  $d$ -spaces. In Example 3.6 and Example 3.7 below, we will show that the converses are not true, respectively.

**Example 3.6** Consider set  $N$  of natural numbers. Let  $X = (N, \tau_{cof})$  be the space  $N$  equipped with the co-finite topology (the empty set and the complements of finite subsets of  $N$  are open sets). Then

- (a)  $\Gamma(X) = \{\emptyset, N\} \cup N^{<\omega}$ ,  $X$  is  $T_1$ , hence it is a  $d$ -space.
- (b)  $X$  is  $s$ -well-filtered since a subset in  $T_1$  spaces is strongly compact iff it is finite.
- (c)  $K(X) = 2^N \setminus \emptyset$ .
- (d)  $\text{RD}(X) = \{N\} \cup \{\{n\} \mid n \in N\}$ .
- (e)  $X$  is not well-filtered.

**Example 3.7** (Johnstone space) Recall the dcpo constructed by Johnstone in [5], which is defined as  $\mathbb{J} = N \times (N \cup \{\infty\})$ , with the order defined by  $(j, k) \leq (m, n)$  iff  $j = m$  and  $k \leq n$  or  $n = \infty$  and  $k \leq m$ . Let  $X = (\mathbb{J}, \tau_\sigma)$ . Then

- (a)  $\mathbb{J}$  is a dcpo, thus  $X$  is a  $d$ -space.
- (b)  $Q(X) = Q_s(X)$ .
- (c)  $X$  is not well-filtered.
- (d)  $X$  is not  $s$ -well-filtered.

Using the equivalent classes of directed subsets, Ershov introduced one way to get  $d$ -completions of  $T_0$  spaces in [3]. Inspired by his method, Liu, Li and Wu presented one way to get well-filtered reflections of  $T_0$  spaces using the equivalent classes of KF-subsets in [10]. Now for a  $K$ -subset system  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$ , we use the equivalent classes of  $k$ -Rudin sets to construct the  $k$ -well-filtered reflections of  $T_0$  spaces. Let  $(X, \tau)$  be a  $T_0$  space. Consider an equivalence relation  $\sim$  on  $K^R(X)$  which is defined as follows:

$$A_0 \sim A_1 \text{ if and only if } A_0 \cap U \neq \emptyset \text{ is equivalent to } A_1 \cap U \neq \emptyset \text{ for any } U \in \tau$$

where  $A_0, A_1 \in K^R(X)$ . Note that  $A_0 \sim A_1$  if and only if  $\text{cl}_X(A_0) = \text{cl}_X(A_1)$ . Let

$$[A] = \{A' \in K^R(X) \mid A \sim A'\}, A \in K^R(X),$$

$$\begin{aligned} K(X) &= \{[A] \mid A \in K^R(X)\}, \\ U^* &= \{[A] \mid A \cap U \neq \emptyset\}, U \in \tau, \\ \tau^* &= \{U^* \mid U \in \tau\}. \end{aligned}$$

Then  $\tau^*$  is a topology. Moreover,  $(K(X), \tau^*)$  is a  $T_0$  space. For  $Q_k(X) = S^u(X)$ , we denote  $K(X)$  by  $D(X)$ . For  $Q_k(X) = Q(X)$ , we denote  $K(X)$  by  $KF(X)$ .

**Lemma 3.8** Let  $Q_k : \mathbf{Top}_0 \rightarrow \mathbf{Set}$  be a  $K$ -subset system,  $(X, \tau)$  a  $T_0$  space and  $\lambda : X \rightarrow K(X)$  the map defined by  $\lambda(x) = [\{x\}]$ . Then the map  $\lambda$  is a homeomorphic embedding.

**Lemma 3.9** Let  $Q_k : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$  be a  $K$ -subset system and  $(X, \tau)$  a  $T_0$  space. Then the following are equivalent:

1.  $X$  is a  $k$ -well-filtered space.
2.  $K(X) \cong X$  (under the map  $\lambda$ ).

Let  $Q_k : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$  be a  $K$ -subset system and  $(X, \tau)$  a  $T_0$  space. Suppose that  $Y$  is a well-filtered space that has  $X$  as a subspace. Since  $Y$  is well-filtered, it is  $k$ -well-filtered. By Lemma 3.9, we have  $K(Y) \cong Y$ . In general, we can consider  $K_\gamma(X)$  as a subspace of  $K_\beta(X)$  in the sense of embedding mappings for all ordinals  $\gamma \leq \beta$ . The transfinite sequence of extensions is constructed as follows:

- (1)  $K_0(X) = X$ ,
- (2)  $K_{\beta+1}(X) = K(K_\beta(X))$ ,
- (3)  $K_\beta(X) = \bigcup_{\gamma < \beta} K_\gamma(X)$  if  $\beta$  is a limit ordinal.

By [2] and [12], we have the following similar results.

**Theorem 3.10** For a  $K$ -subset system  $Q_k : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$  and a  $T_0$  space  $(X, \tau)$ , the  $k$ -well-filterification of  $X$  exists; i.e., there exists an ordinal  $\alpha$  such that  $H_k(X) = K_\alpha(X) \cong K_{\alpha+1}(X)$ .

**Proof.** The proof is similar to the method of constructing the  $d$ -completion of  $T_0$  spaces in [2] and the method of constructing the well-filterification of  $T_0$  spaces in [12].  $\square$

**Definition 3.11** Let  $Q_k : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$  be a  $K$ -subset system and  $(X, \tau)$  a  $T_0$  space. The  $k$ -rank of  $X$  is the least ordinal  $\alpha$  such that  $K_\alpha(X) \cong K_{\alpha+1}(X)$ . We denote the  $k$ -rank of a space  $X$  by  $\text{rank}_k(X)$ .

Similarly, the  $d$ -rank of  $X$  is the least ordinal  $\alpha$  such that  $D_\alpha(X) \cong D_{\alpha+1}(X)$ , it is denoted by  $\text{rank}_d(X)$  in [3]. The  $wf$ -rank of  $X$  is the least ordinal  $\alpha$  such that  $KF_\alpha(X) \cong KF_{\alpha+1}(X)$ , it is denoted by  $\text{rank}_{wf}(X)$  in [8].

## 4 $\alpha^k$ -special spaces

For a  $K$ -subset system  $Q_k : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ , in Theorem 3.10, there exists an ordinal  $\alpha$  such that  $\text{rank}_k(X) = \alpha$  for a  $T_0$  space  $X$ . Conversely, for any given ordinal  $\alpha$  it is natural to ask whether there exists a  $T_0$  space  $X$  such that  $\text{rank}_k(X) = \alpha$ . In this section, we prove that for any given ordinal  $\alpha$ , there exists a  $T_0$  space  $X$  such that  $\text{rank}_k(X) = \alpha$ .

**Definition 4.1** Let  $Q_k : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$  be a  $K$ -subset system and  $(X, \tau)$  a  $T_0$  space. For an ordinal  $\alpha$ ,  $X$  is called  $\alpha^k$ -special if the following conditions are satisfied:

- (1)  $\text{rank}_k(X) = \alpha$ ;
- (2)  $\alpha$  is the least ordinal for which  $K_\alpha(X)$  has a greatest element.

$X$  is called  $\alpha^d$ -special (resp.,  $\alpha^{wf}$ -special), similarly, see [3] and [8], respectively.

**Remark 4.2** If  $X$  is a  $\alpha^k$ -special space, then  $\alpha$  is not a limit ordinal.

**Proof.** In fact, let  $(X, \tau)$  be a  $\alpha^k$ -special space. Suppose that  $\alpha$  is a limit ordinal. Then  $K_\alpha(X) = \bigcup_{\beta < \alpha} K_\beta(X)$ . By Definition 4.1,  $K_\alpha(X)$  has a greatest element. Hence, there exists  $\beta < \alpha$  such that  $K_\beta(X)$  has a greatest element, which is a contradiction. So  $\alpha$  will not be a limit ordinal.  $\square$

**Lemma 4.3** For any nonlimit ordinal  $\alpha$ , every  $\alpha^k$ -special space is irreducible.

**Proof.** The proof is similar to Lemma 3.3 in [3].  $\square$

Recall the following construction in [3]. For topological spaces  $X$  and  $Y_x$ ,  $x \in X$ , let

$$Z = \bigcup_{x \in X} Y_x \times \{x\},$$

$$\tau = \{U \subseteq Z \mid (U)_x \in \tau(Y_x) \text{ for any } x \in X \text{ and } (U)_X \in \tau(X)\},$$

where  $(U)_x = \{y \in Y_x \mid (y, x) \in U\}$  for any  $x \in X$  and  $(U)_X = \{x \in X \mid (U)_x \neq \emptyset\}$ .

**Lemma 4.4** ([3]) *Let  $X$  be a  $T_0$  space and  $Y_x$  an irreducible  $T_0$  space for every  $x \in X$ . Then*

- (1)  $\tau$  is a  $T_0$  separable topology on  $Z$ .
- (2) The map  $y \mapsto (y, x)$  determines a homeomorphic embedding of  $Y_x$  in  $Z$  for any  $x \in X$ .
- (3) If the space  $X$  is irreducible, then the space  $Z$  is also irreducible.

For any subset  $A \subseteq Z$ , put  $\tilde{X} = \{x \in X \mid Y_x \text{ has the greatest element } \top_x\}$  with the induced topology of  $X$ . Define

$$(A)_x = \{y \in Y_x \mid (y, x) \in A\} \text{ for any } x \in X,$$

$$(A)_X = \{x \in X \mid (A)_x \neq \emptyset\},$$

$$A_* = \{x \in \tilde{X} \mid (\top_x, x) \in A\}.$$

And the space  $(Z, \tau)$  is also denoted by  $\sum_X Y_x$ .

**Lemma 4.5** ([3]) *Let  $X$  be a  $T_0$  space,  $Y_x$  an irreducible  $T_0$  space for any  $x \in X$ , and  $Z = \sum_X Y_x$ . For all*

*$(y_0, x_0), (y_1, x_1) \in Z$ , we have  $(y_0, x_0) \leq (y_1, x_1)$  if and only if the following two alternatives hold:*

- (1)  $x_0 = x_1$  and  $y_0 \leq_{Y_{x_0}} y_1$ ;
- (2)  $x_0 <_X x_1$  and  $y_1 = \top_{x_1}$  is the greatest element in  $Y_{x_1}$ .

**Lemma 4.6** ([3]) *Let  $X$  be a  $T_0$  space,  $Y_x$  an irreducible  $T_0$  space for any  $x \in X$ , and  $Z = \sum_X Y_x$ . Then*

*an arbitrary set  $S' \in \mathcal{D}(Z)$  contains a cofinal subset  $S \subseteq S'$  having one of the following forms:*

- (i)  $S = \{(y, x) \mid y \in (S)_x\}$  for some fixed  $x \in X$  and  $(S)_x \in \mathcal{D}(Y_x)$ ;
- (ii)  $S = \{(\top_x, x) \mid x \in S_*\}$  for some  $S_* \in \mathcal{D}(\tilde{X})$ .

For any irreducible topological space  $Y$ , put

$$Y^\top = \begin{cases} Y, & \text{if } Y \text{ has a greatest element,} \\ \langle Y \cup \{\top\}, \tau(Y)^\top \rangle, & \text{otherwise,} \end{cases}$$

where  $\tau(Y)^\top = \{U \cup \{\top\} \mid \emptyset \neq U \in \tau(Y)\} \cup \{\emptyset\}$ . It is easy to see that for any irreducible  $T_0$  space  $Y$ ,  $Y^\top$  is also a  $T_0$  space and has a greatest element. Let

$$X' = \{[\{x\}] \mid x \in X\} \cup D(\tilde{X}) \subseteq D(X).$$

Then  $X'$  with the induced topology is a subspace of  $D(X)$  and the space  $D(Y_x)$  is irreducible for any  $x \in X$  from [3]. Moreover, for any  $x' \in X'$ , we define

$$Y'_{x'} = \begin{cases} D(Y_x), & \text{if } x' = [\{x\}] \text{ for some } x \in X, \\ \top, & \text{otherwise,} \end{cases}$$

where  $\top = \langle \{\top\}, \{\emptyset, \{\top\}\} \rangle$ . Then we have the following theorem.



**Theorem 4.7** ([3]) *Let  $X$  be a  $T_0$  space,  $Y_x$  an irreducible  $T_0$  space for any  $x \in X$ , and  $Z = \sum_X Y_x$ . Then the spaces  $D(Z)$  and  $Z' = \sum_{X'} Y_{X'}$  are homeomorphic.*

For any ordinal  $\alpha > 0$ , consider the irreducible  $T_0$  space

$$\mathbb{O}_\alpha = \langle \downarrow\alpha \setminus \{\alpha\}, \{\emptyset\} \cup \{\uparrow\beta \mid \beta < \alpha \text{ is not a limit ordinal}\} \rangle.$$

**Proposition 4.8** ([3]) *Let  $\alpha > 0$  be an ordinal.*

- (i) *If  $\alpha$  is a limit ordinal, then  $\mathbb{H}_d(\mathbb{O}_\alpha) = \mathbb{O}_\alpha^\top = D(\mathbb{O}_\alpha)$ , i.e., the space  $\mathbb{O}_\alpha$  is  $1^d$ -special.*
- (ii) *If  $\alpha$  is not a limit ordinal, then  $\mathbb{H}_d(\mathbb{O}_\alpha) = \mathbb{O}_\alpha$ , i.e., the  $d$ -rank of  $\mathbb{O}_\alpha$  is equal to 0.*
- (iii) *If  $\alpha$  is a limit ordinal,  $\gamma$  is not a limit ordinal, a  $T_0$  space  $Y_\beta$  is  $\gamma^d$ -special for any  $\beta < \alpha$  and the space  $Z = \sum_{\mathbb{O}_\alpha} Y_\beta$ , then the spaces  $D_\delta(Z) \cong \sum_{\mathbb{O}_\alpha} D_\delta(Y_\beta)$  for any ordinal  $\delta \leq \gamma$ .*
- (iv) *If  $\alpha$  is a limit ordinal,  $\gamma$  is not a limit ordinal and a  $T_0$  space  $Y_\beta$  is  $\gamma^d$ -special for any  $\beta < \alpha$ , then the space  $Z = \sum_{\mathbb{O}_\alpha} Y_\beta$  is  $(\gamma + 1)^d$ -special.*
- (v) *If  $Y$  is an  $(\alpha + 1)^d$ -special for some ordinal  $\alpha$ , then  $D_\beta(Y^\top) \cong D_\beta(Y)^\top$  for any  $\beta \leq \alpha$  and  $D_{\alpha+1}(Y^\top) \cong D_{\alpha+1}(Y) = \mathbb{H}_d(Y)$ .*
- (vi) *If  $\alpha$  is a limit ordinal, and a  $T_0$  space  $Y_\beta$  is  $(\beta + 1)^d$ -special for any  $\beta < \alpha$ , then the space  $Z = \sum_{\mathbb{O}_\alpha} Y_\beta$  is  $(\alpha + 1)^d$ -special and the  $d$ -rank of a space  $Z^\top$  is equal to  $\alpha$ .*

**Lemma 4.9** *Let  $\mathbb{N} = (N, \tau_\sigma)$  denote the set  $N$  of natural numbers endowed with the Scott topology. Then  $\mathbb{N}$  is  $1^d$ -special.*

**Proof.** This directly follows from (i) of Proposition 4.8. □

**Lemma 4.10** *Let  $\alpha > 0$  be an ordinal.*

- (1) *If  $\alpha$  is not a limit ordinal and a  $T_0$  space  $X_n$  is  $\alpha^d$ -special for any  $n \in N$ , then the space  $Z = \sum_{\mathbb{N}} X_n$  is  $(\alpha + 1)^d$ -special.*
- (2) *If  $\alpha$  is a limit ordinal and a  $T_0$  space  $X_n$  is  $(\bar{\alpha} + n)^d$ -special for any  $n \in N$ , then the space  $Z = \sum_{\mathbb{N}} X_n$  is  $(\alpha + 1)^d$ -special and the  $d$ -rank of the space  $Z^\top$  is equal to  $\alpha$ , where  $\bar{\alpha} = 0$  if  $\alpha = \omega$ , otherwise,  $\bar{\alpha}$  denotes the largest limit ordinal less than  $\alpha$ .*

**Proof.**

- (1) It follows directly from (iv) of Proposition 4.8.
- (2) First we prove that the spaces  $D_\delta(Z)$  and  $\sum_{\mathbb{N}} W_n^\delta$  are homeomorphic for every ordinal  $\delta < \alpha$ , where

$$W_n^\delta = \begin{cases} D_\delta(X_n), & \text{if } \delta < \bar{\alpha} + n, \\ \mathbb{H}_d(X_n), & \text{if } \bar{\alpha} + n \leq \delta < \alpha. \end{cases}$$

We use induction on  $\delta$ . For  $\delta = 0$ , the statement follows from the definition of space  $Z$ .

Let  $\delta$  be an ordinal such that  $\delta + 1 < \alpha$ , and suppose that  $D_\delta(Z) \cong \sum_{\mathbb{N}} W_n^\delta$ . Then the space  $D_\delta(X_n)$  does not contain a greatest element for the arbitrary  $n \in N$  such that  $\delta < \bar{\alpha} + n$ . Note that  $\tilde{N} = \{n \in N \mid \bar{\alpha} + n \leq \delta\}$  is a finite subset in  $N$ , hence  $N' = \{\{n\} \mid n \in N\} \cup D(\tilde{N}) \cong \mathbb{N}$ . By

Theorem 4.7, we have

$$D_{\delta+1}(Z) = D(D_\delta(Z)) \cong D\left(\sum_{\mathbb{N}} W_n^\delta\right) \cong \sum_{\mathbb{N}} D(W_n^\delta) = \sum_{\mathbb{N}} W_n^{\delta+1}.$$

Suppose now that  $\delta < \alpha$  is a limit ordinal and  $D_\beta(Z) \cong \sum_{\mathbb{N}} W_n^\beta$  for any ordinal  $\beta < \delta$ . By Theorem 4.7, we get

$$D_\delta(Z) = \bigcup_{\beta < \delta} D_\beta(Z) \cong \bigcup_{\beta < \delta} \sum_{\mathbb{N}} W_n^\beta \cong \sum_{\mathbb{N}} \bigcup_{\beta < \delta} W_n^\beta = \sum_{\mathbb{N}} W_n^\delta.$$

Thus by induction, we have  $D_\delta(Z) \cong \sum_{\mathbb{N}} W_n^\delta$  for any ordinal  $\delta < \alpha$ .

Therefore,

$$D_\alpha(Z) = \bigcup_{\delta < \alpha} D_\delta(Z) \cong \bigcup_{\delta < \alpha} \sum_{\mathbb{N}} W_n^\delta \cong \sum_{\mathbb{N}} \bigcup_{\delta < \alpha} W_n^\delta = \sum_{\mathbb{N}} W_n^\alpha \cong \sum_{\mathbb{N}} H_d(X_n).$$

Moreover, for any  $n \in N$ , the space  $H_d(X_n)$  has a greatest element, which implies that  $\tilde{N} = N$ . Hence  $\mathbb{N}' \cong \mathbb{N}^\top$ . In the view of Lemma 4.5 and Theorem 4.7, we obtain

$$D_{\alpha+1}(Z) = D(D_\alpha(Z)) \cong D\left(\sum_{\mathbb{N}} H_d(X_n)\right) \cong \sum_{\mathbb{N}^\top} X'_{n'} \cong \left(\sum_{\mathbb{N}} H_d(X_n)\right)^\top,$$

where

$$X'_{n'} = \begin{cases} H_d(X_{n'}), & \text{if } n' \in N, \\ \top, & \text{if } n' = \top, \end{cases}$$

and

$$D_{\alpha+2}(Z) = D(D_{\alpha+1}(Z)) \cong D\left(\sum_{\mathbb{N}^\top} X'_{n'}\right) \cong \sum_{\mathbb{N}^\top} D(X'_{n'}) \cong \sum_{\mathbb{N}^\top} X'_{n'} \cong D_{\alpha+1}(Z).$$

Again by Lemma 4.5, the space  $D_\beta(Z)$  has not a greatest element for any ordinal  $\beta \leq \alpha$ . Therefore, by virtue of Definition 4.1, the space  $Z$  is  $(\alpha + 1)^d$ -special.

For  $Z^\top$ , first, we claim that the spaces  $D_\delta(Z^\top) \cong \sum_{\mathbb{N}^\top} W_{n'}^\delta$  for the arbitrary ordinal  $\delta \leq \alpha$ , where

$$W_{n'}^\delta = \begin{cases} D_\delta(X_{n'}), & \text{if } \delta < \bar{\alpha} + n' < \alpha, \\ H_d(X_{n'}), & \text{if } \bar{\alpha} + n' \leq \delta \leq \alpha, \\ \top, & \text{if } n' = \top, \end{cases}$$

By the part (v) of Proposition 4.8, we get

$$D_\delta(Z^\top) \cong (D_\delta(Z))^\top \cong \left(\sum_{\mathbb{N}} W_n^\delta\right)^\top \cong \sum_{\mathbb{N}^\top} W_{n'}^\delta$$

for every ordinal  $\delta \leq \alpha$ . This implies that

$$D_\alpha(Z^\top) \cong \sum_{\mathbb{N}^\top} W_{n'}^\alpha \cong \left(\sum_{\mathbb{N}} H_d(X_n)\right)^\top \cong D_{\alpha+1}(Z),$$

which is a  $d$ -space by the above proof. Therefore,  $D_\alpha(Z^\top) \cong D_{\alpha+1}(Z^\top)$ .



Next, we claim that  $D_\delta(Z^\top)$  is not a  $d$ -space for any ordinal  $\delta < \alpha$ . Assume that there exists an ordinal  $\delta < \alpha$  such that  $D_\delta(Z^\top)$  is a  $d$ -space. Then by Lemma 3.9,  $D_\delta(Z^\top) \cong D_\alpha(Z^\top)$ . However, from the preceding discussion, we have that the spaces  $D_\delta(Z^\top) \cong \sum_{\mathbb{N}^\top} W_{n'}^\delta$  for the arbitrary ordinal  $\delta < \alpha$ . Note that there are at most finitely many  $W_{n'}^\delta$ 's are  $d$ -spaces. Furthermore, for  $\delta < \bar{\alpha} + n' < \alpha$ ,  $W_{n'}^\delta = D_\delta(X_{n'})$  is not a  $d$ -space and  $W_{n'}^\alpha = H_d(X_{n'})$  is a  $d$ -space, which implies that  $W_{n'}^\delta$  and  $W_{n'}^\alpha$  are not homeomorphic. Hence,  $D_\delta(Z^\top)$  and  $D_\alpha(Z^\top)$  are not homeomorphic, which is a contradiction. So the  $d$ -rank of the space  $Z^\top$  is equal to  $\alpha$ . □

For the  $wf$ -rank of a  $T_0$  space, we have the following similar results.

**Lemma 4.11** [11] *Let  $X$  be a  $T_0$  space,  $Y_x$  an irreducible  $T_0$  space for any  $x \in X$ , and  $Z = \sum_X Y_x$ . Then an arbitrary set  $A' \in \overline{KF}(Z)$  contains a subset  $A \subseteq A'$  such that  $A \sim A'$  having one of the following forms:*

- (1) *there exists an element  $x \in X$  such that  $A \subseteq Y_x \times \{x\}$  and  $(A)_x \in \overline{KF}(Y_x)$ ;*
- (2)  *$A = \{(\top_x, x) \mid x \in A_*\}$  for some  $A_* \in \overline{KF}(\tilde{X})$ , where  $\tilde{X} = \{x \in X \mid Y_x \text{ has a greatest element}\}$ .*

**Theorem 4.12** *Let  $X$  be a  $T_0$  space,  $Y_x$  an irreducible  $T_0$  space for any  $x \in X$ , and  $Z = \sum_X Y_x$ . Then the spaces  $KF(Z)$  and  $Z' = \sum_{X'} Y'_{x'}$  are homeomorphic, where*

$$X' = \{[\{x\}] \mid x \in X\} \cup KF(\tilde{X}) \subseteq KF(X),$$

and for any  $x' \in X'$

$$Y'_{x'} = \begin{cases} KF(Y_x), & \text{if } x' = [\{x\}] \text{ for some } x \in X, \\ \top, & \text{otherwise, where } \top = \langle \{\top\}, \{\emptyset, \{\top\}\} \rangle. \end{cases}$$

**Lemma 4.13**  $(N, \tau_\sigma)$  is  $1^{wf}$ -special.

**Lemma 4.14** Let  $\alpha > 0$  be an ordinal.

- (1) *If  $\alpha$  is not a limit ordinal and a  $T_0$  space  $X_n$  is  $\alpha^{wf}$ -special for any  $n \in N$ , then the space  $Z = \sum_{\mathbb{N}} X_n$  is  $(\alpha + 1)^{wf}$ -special.*
- (2) *If  $\alpha$  is a limit ordinal and a  $T_0$  space  $X_n$  is  $(\bar{\alpha} + n)^{wf}$ -special for any  $n \in N$ , then the space  $Z = \sum_{\mathbb{N}} X_n$  is  $(\alpha + 1)^{wf}$ -special and the  $wf$ -rank of the space  $Z^\top$  is equal to  $\alpha$ , where  $\bar{\alpha} = 0$  if  $\alpha = \omega$ , otherwise,  $\bar{\alpha}$  denotes the largest limit ordinal less than  $\alpha$ .*

Next, let  $Q_k$  be a  $K$ -subset system and  $X$  a  $T_0$  space. We denote

$$\overline{D}(X) = \{A \subseteq X \mid \text{there exists a directed subset } D \text{ in } X \text{ such that } \overline{A} = \overline{D}\}.$$

For the  $k$ -rank of  $X$ , we deduce the following results.

**Lemma 4.15** For  $\mathbb{N} = (N, \tau_\sigma)$ , the following statements hold:

- (1)  $\mathbb{N}$  is  $1^d$ -special and  $1^{wf}$ -special.
- (2)  $\overline{D}(\mathbb{N}) = \overline{KF}(\mathbb{N})$  and  $D(\mathbb{N}) = K(\mathbb{N}) = KF(\mathbb{N})$ .
- (3)  $\mathbb{N}$  is  $1^k$ -special.

**Lemma 4.16** *If  $\alpha$  is not a limit ordinal and a  $T_0$  space  $X_n$  satisfies the following conditions.*

(1)  $X_n$  is  $\alpha^d$ -special and  $\alpha^{wf}$ -special,  
(2)  $\overline{D}(K_m(X_n)) = \overline{KF}(K_m(X_n))$  and  $D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n)$  for  $0 \leq m < \alpha$ ,  
for any  $n \in N$ , then the space  $Z = \sum_{\mathbb{N}} X_n$  satisfies:

- (1)  $Z$  is  $(\alpha + 1)^d$ -special and  $(\alpha + 1)^{wf}$ -special.  
(2)  $\overline{D}(K_m(Z)) = \overline{KF}(K_m(Z))$  and  $D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z)$  for  $0 \leq m < \alpha + 1$ .  
(3)  $Z$  is  $(\alpha + 1)^k$ -special.

**Proof.** For (1), this directly follows from Lemma 4.10 (1) and Lemma 4.14 (1).

For (2), the proof is by induction on  $m$ .

Basic steps. For  $m = 0$ , obviously,  $\overline{D}(Z) \subseteq \overline{KF}(Z)$ . Conversely, let  $A \in \overline{KF}(Z)$ . From Lemma 4.11, there exists a subset  $A' \subseteq A$  that  $A' \sim A$  and  $A'$  satisfies Type (i) in Lemma 4.11. This means that there exists  $n \in N$  such that  $A' \subseteq X_n \times \{n\}$  and  $(A')_n \in \overline{KF}(X_n) = \overline{D}(X_n)$ . By the definition of  $\overline{D}(X_n)$ , there is a directed subset  $D$  in  $X_n$  such that  $\text{cl}_{X_n}((A')_n) = \text{cl}_{X_n}(D)$ . We claim that  $\text{cl}_Z(A') = \text{cl}_Z(D \times \{n\})$ . For  $(y, x) \in \text{cl}_Z(A')$ , let  $U$  be an open neighbourhood of  $(y, x)$ . Then we have that  $U \cap A' \neq \emptyset$ . This implies that  $(U)_n \cap (A')_n \neq \emptyset$ . Since  $U$  is open in  $Z$ , we have  $(U)_n \in \tau(X_n)$ . By  $\text{cl}_{X_n}((A')_n) = \text{cl}_{X_n}(D)$ , we have  $(U)_n \cap D \neq \emptyset$ , that is  $U \cap (D \times \{n\}) \neq \emptyset$ . Hence  $(y, x) \in \text{cl}_Z(D \times \{n\})$ . The opposite direction is similar to prove. So  $A \in \overline{D}(Z)$ . That is  $\overline{D}(Z) = \overline{KF}(Z)$ , which implies that  $\overline{D}(Z) = \overline{K}(Z) = \overline{KF}(Z)$ . Therefore,  $D(Z) = K(Z) = KF(Z)$ .

Inductive steps. There are two cases to consider:

Case 1. Let  $m$  be an ordinal such that  $m + 1 < \alpha + 1$ . Assume that  $\overline{D}(K_m(Z)) = \overline{KF}(K_m(Z))$  and  $D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z)$ , by Lemma 4.8 (iii), we have

$$D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z) \cong \sum_{\mathbb{N}} D_{m+1}(X_n).$$

To prove  $\overline{D}(K_{m+1}(Z)) = \overline{KF}(K_{m+1}(Z))$ , it is enough to show  $\overline{D}(\sum_{\mathbb{N}} D_{m+1}(X_n)) = \overline{KF}(\sum_{\mathbb{N}} D_{m+1}(X_n))$ .

Clearly,  $\overline{D}(\sum_{\mathbb{N}} D_{m+1}(X_n)) \subseteq \overline{KF}(\sum_{\mathbb{N}} D_{m+1}(X_n))$ . Conversely, for any  $A \in \overline{KF}(\sum_{\mathbb{N}} D_{m+1}(X_n))$ , by Lemma 4.11, there exists a subset  $A' \subseteq A$  such that  $A' \sim A$ . Two options are possible:

Case 1.1.  $A'$  is Type (i) in Lemma 4.11. This implies that there exists  $n \in N$  such that  $A' \subseteq D_{m+1}(X_n) \times \{n\}$  and  $(A')_n \in \overline{KF}(D_{m+1}(X_n))$ . By the condition (2) of  $X_n$ , we get

$$D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n) \text{ and } \overline{KF}(K_{m+1}(X_n)) = \overline{D}(K_{m+1}(X_n)).$$

Hence,  $(A')_n \in \overline{KF}(D_{m+1}(X_n)) = \overline{D}(D_{m+1}(X_n))$ . So  $A' \in \overline{D}(\sum_{\mathbb{N}} (D_{m+1}(X_n)))$ . This implies that  $A \in \overline{D}(\sum_{\mathbb{N}} (D_{m+1}(X_n)))$ .

Case 1.2.  $A'$  is Type (ii) in Lemma 4.11. This means that there exists  $A_* \in \overline{KF}(\tilde{N})$  such that  $A' = \{(\top_n, n) \mid n \in A_*\}$ . Note that  $m + 1 = \alpha$ , then  $\sum_{\mathbb{N}} D_{m+1}(X_n) \cong \sum_{\mathbb{N}} H_d(X_n) \cong \sum_{\mathbb{N}} H_{wf}(X_n)$ . So  $\tilde{N} = \mathbb{N}$ . Hence,  $\overline{KF}(\tilde{N}) = \overline{D}(\tilde{N})$ , which implies that  $A' \in \overline{D}(\sum_{\mathbb{N}} D_{m+1}(X_n))$ . Therefore,  $A \in \overline{D}(\sum_{\mathbb{N}} D_{m+1}(X_n))$ .

In any case, we have  $\overline{KF}(\sum_{\mathbb{N}} D_{m+1}(X_n)) \subseteq \overline{D}(\sum_{\mathbb{N}} D_{m+1}(X_n))$ . So  $\overline{D}(K_{m+1}(Z)) = \overline{KF}(K_{m+1}(Z))$ . This implies that  $D_{m+2}(Z) = K_{m+2}(Z) = KF_{m+2}(Z)$ .

Case 2. Suppose that  $m < \alpha + 1$  is a limit ordinal and the required statement holds for any  $\delta < m$ . Then

$$K_m(Z) = \bigcup_{\delta < m} K_\delta(Z) \cong \bigcup_{\delta < m} D_\delta(Z) \cong \bigcup_{\delta < m} \sum_{\mathbb{N}} D_\delta(X_n) \cong \sum_{\mathbb{N}} \bigcup_{\delta < m} D_\delta(X_n) \cong \sum_{\mathbb{N}} D_m(X_n).$$

Now it is enough to show that  $\overline{KF}(\sum_{\mathbb{N}} D_m(X_n)) = \overline{D}(\sum_{\mathbb{N}} D_m(X_n))$ . Repeat the proof method of Case 1, we get that  $\overline{D}(K_m(Z)) = \overline{KF}(K_m(Z))$ . So  $\overline{D}(K_m(Z)) = \overline{K}(K_m(Z)) = \overline{KF}(K_m(Z))$ . Hence,  $D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z)$ .

For (3), by (2), let  $m = \alpha$ . We get

$$D_{\alpha+1}(Z) = K_{\alpha+1}(Z) = KF_{\alpha+1}(Z).$$

Since  $Z$  is  $(\alpha + 1)^d$ -special and  $(\alpha + 1)^{wf}$ -special, for any ordinal  $\delta < \alpha + 1$ ,  $K_\delta(Z)$  is not a  $d$ -space and  $K_{\alpha+1}(Z)$  is well-filtered. Therefore,  $K_\delta(Z)$  is not  $k$ -well-filtered and  $K_{\alpha+1}(Z)$  is  $k$ -well-filtered. Then  $Z$  is  $(\alpha + 1)^k$ -special.  $\square$

**Lemma 4.17** *If  $\alpha$  is a limit ordinal and a  $T_0$  space  $X_n$  satisfies the following conditions.*

(1)  $X_n$  is  $(\bar{\alpha} + n)^d$ -special and  $(\bar{\alpha} + n)^{wf}$ -special,

(2)  $\overline{D}(K_m(X_n)) = \overline{KF}(K_m(X_n))$  and  $D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n)$  for  $0 \leq m < \bar{\alpha} + n$ ,

for any  $n \in \mathbb{N}$ , then for the space  $Z = \sum_{\mathbb{N}} X_n$ , we have the following conclusions.

(1)  $Z$  is  $(\alpha + 1)^d$ -special and  $(\alpha + 1)^{wf}$ -special.

(2)  $\overline{D}(K_m(Z)) = \overline{KF}(K_m(Z))$  and  $D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z)$  for  $0 \leq m < \alpha + 1$ .

(3)  $Z$  is  $(\alpha + 1)^k$ -special.

Moreover, for the space  $Z^\top$ , the following results hold.

(1)  $\text{rank}_d(Z^\top) = \text{rank}_{wf}(Z^\top) = \alpha$ ,

(2)  $\overline{D}(K_m(Z^\top)) = \overline{KF}(K_m(Z^\top))$  and  $D_{m+1}(Z^\top) = K_{m+1}(Z^\top) = KF_{m+1}(Z^\top)$  for  $0 \leq m < \alpha$ ,

(3)  $\text{rank}_k(Z^\top) = \alpha$ .

**Proof.** First, we consider the space  $Z = \sum_{\mathbb{N}} X_n$ .

For (1), it follows directly from Lemma 4.10 (2) and Lemma 4.14 (2).

For (2), we proceed by induction. For  $m = 0$ , the statement follows from the proof of Lemma 4.16. Let  $m$  be an ordinal such that  $m + 1 < \alpha + 1$ . Assume that  $\overline{D}(K_m(Z)) = \overline{KF}(K_m(Z))$  and  $D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z)$ , by the proof of Lemma 4.10 (2), we derive

$$D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z) \cong \sum_{\mathbb{N}} W_n^{m+1}$$

where

$$W_n^{m+1} = \begin{cases} D_{m+1}(X_n), & \text{if } m + 1 < \bar{\alpha} + n, \\ H_d(X_n), & \text{if } \bar{\alpha} + n \leq m + 1 < \alpha + 1. \end{cases}$$

To prove  $\overline{D}(K_{m+1}(Z)) = \overline{KF}(K_{m+1}(Z))$ , it suffices to show that  $\overline{D}(\sum_{\mathbb{N}} W_n^{m+1}) = \overline{KF}(\sum_{\mathbb{N}} W_n^{m+1})$ . Clearly,  $\overline{D}(\sum_{\mathbb{N}} W_n^{m+1}) \subseteq \overline{KF}(\sum_{\mathbb{N}} W_n^{m+1})$ . For any  $A \in \overline{KF}(\sum_{\mathbb{N}} W_n^{m+1})$ , by Lemma 4.11, there exists a subset  $A' \subseteq A$  such that  $A' \sim A$ . There are two cases to consider:

Case 1.  $A'$  is Type (i) in Lemma 4.11. This implies that there exists  $n \in \mathbb{N}$  such that  $A' \subseteq W_n^{m+1} \times \{n\}$  and  $(A')_n \in \overline{KF}(W_n^{m+1})$ . Again there are two cases to consider:

Case 1.1. If  $m + 1 < \bar{\alpha} + n$ , then  $W_n^{m+1} = D_{m+1}(X_n)$ . By the condition (2) of  $X_n$ , we have

$$D_{m+1}(X_n) = K_{m+1}(X_n) = KF_{m+1}(X_n) \text{ and } \overline{KF}(K_{m+1}(X_n)) = \overline{D}(K_{m+1}(X_n)).$$

Hence,  $(A')_n \in \overline{KF}(D_{m+1}(X_n)) = \overline{D}(D_{m+1}(X_n))$ . So  $A' \in \overline{D}(\sum_{\mathbb{N}} W_n^{m+1})$ . This implies that  $A \in \overline{D}(\sum_{\mathbb{N}} W_n^{m+1})$ .

Case 1.2. If  $\bar{\alpha} + n \leq m+1 < \alpha+1$ , then  $W_n^{m+1} = H_{wf}(X_n) = H_d(X_n)$ . Hence,  $\overline{KF}(W_n^{m+1}) = \overline{D}(W_n^{m+1})$ . It is straightforward to check that  $A \in \overline{D}(\sum_{\mathbb{N}} W_n^{m+1})$ .

Case 2.  $A'$  is Type (ii) in Lemma 4.11. This means that there exists  $A_* \in \overline{KF}(\tilde{\mathbb{N}})$  such that  $A' = \{(\top_n, n) \mid n \in A_*\}$ . Note that there are at most finitely many  $W_n^{m+1}$ 's which have a greatest element; thus  $\overline{KF}(\tilde{\mathbb{N}}) = \overline{D}(\tilde{\mathbb{N}})$ . This implies that  $A' \in \overline{D}(\sum_{\mathbb{N}} W_n^{m+1})$ . Therefore,  $A \in \overline{D}(\sum_{\mathbb{N}} W_n^{m+1})$ .

In any case, we have that  $\overline{KF}(\sum_{\mathbb{N}} W_n^{m+1}) \subseteq \overline{D}(\sum_{\mathbb{N}} W_n^{m+1})$ . So  $\overline{D}(K_{m+1}(Z)) = \overline{KF}(K_{m+1}(Z))$ . This implies that  $D_{m+2}(Z) = K_{m+2}(Z) = KF_{m+2}(Z)$ .

Suppose that  $m < \alpha + 1$  is a limit ordinal and that the required statement holds for any  $\delta < m$ . Then

$$K_m(Z) = \bigcup_{\delta < m} K_\delta(Z) \cong \bigcup_{\delta < m} D_\delta(Z) \cong \bigcup_{\delta < m} \sum_{\mathbb{N}} W_n^\delta \cong \sum_{\mathbb{N}} \bigcup_{\delta < m} W_n^\delta \cong \sum_{\mathbb{N}} W_n^m$$

where

$$W_n^\delta = \begin{cases} D_\delta(X_n), & \text{if } \delta < \bar{\alpha} + n, \\ H_d(X_n), & \text{if } \bar{\alpha} + n \leq \delta < \alpha. \end{cases}$$

Now it is enough to show that  $\overline{KF}(\sum_{\mathbb{N}} W_n^m) = \overline{D}(\sum_{\mathbb{N}} W_n^m)$ . Repeat the above proof method, we get that  $\overline{D}(K_m(Z)) = \overline{KF}(K_m(Z))$ . Hence,

$$\overline{D}(K_m(Z)) = \overline{K}(K_m(Z)) = \overline{KF}(K_m(Z)) \text{ and } D_{m+1}(Z) = K_{m+1}(Z) = KF_{m+1}(Z).$$

For (3), let  $m = \alpha$ . By (2), we deduce

$$D_{\alpha+1}(Z) = K_{\alpha+1}(Z) = KF_{\alpha+1}(Z).$$

Since  $Z$  is  $(\alpha + 1)^d$ -special and  $(\alpha + 1)^{wf}$ -special, we have that for any ordinal  $\delta < \alpha + 1$ ,  $K_\delta(Z)$  is not a  $d$ -space and  $K_{\alpha+1}(Z)$  is well-filtered. Thus  $K_\delta(Z)$  is not  $k$ -well-filtered and  $K_{\alpha+1}(Z)$  is  $k$ -well-filtered. Then  $Z$  is  $(\alpha + 1)^k$ -special.

Next, we analyze the space  $Z^\top$ . For  $Z^\top$ , the statement (1) also follows from Lemma 4.10 (2) and Lemma 4.14 (2).

For (2), The proof is by induction on  $m$ .

Basic steps. For  $m = 0$ , clearly,  $\overline{D}(Z^\top) \subseteq \overline{KF}(Z^\top)$ . Conversely, let  $A \in \overline{KF}(Z^\top)$ . From Lemma 4.11, we have that  $A \in \overline{KF}(Z)$  or  $A \sim \{\top\}$ . It is straightforward to check that  $A \in \overline{D}(Z^\top)$ . Therefore,

$$\overline{D}(Z^\top) = \overline{KF}(Z^\top) \text{ and } D(Z) = K(Z) = KF(Z).$$

Inductive steps. There are two cases to consider:

Case 1. Let  $m$  be an ordinal such that  $m + 1 < \alpha$ . Assume that

$$\overline{D}(K_m(Z^\top)) = \overline{KF}(K_m(Z^\top)) \text{ and } D_{m+1}(Z^\top) = K_{m+1}(Z^\top) = KF_{m+1}(Z^\top).$$

By the proof of Lemma 4.10 (2), we have that  $D_{m+1}(Z^\top) = K_{m+1}(Z^\top) \cong \sum_{\mathbb{N}^\top} W_{n'}^{m+1}$ , where

$$W_{n'}^{m+1} = \begin{cases} D_{m+1}(X_{n'}), & \text{if } m+1 < \bar{\alpha} + n' < \alpha, \\ \text{Hd}(X_{n'}), & \text{if } \bar{\alpha} + n' \leq m+1 < \alpha, \\ \top, & \text{if } n' = \top. \end{cases}$$

To prove  $\overline{D}(K_{m+1}(Z^\top)) = \overline{KF}(K_{m+1}(Z^\top))$ , it is sufficient to show  $\overline{D}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1}) = \overline{KF}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1})$ . Clearly,  $\overline{D}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1}) \subseteq \overline{KF}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1})$ . For any  $A \in \overline{KF}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1})$ , again by Lemma 4.11, we have that  $A \in \overline{D}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1})$ . Therefore,  $\overline{KF}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1}) \subseteq \overline{D}(\sum_{\mathbb{N}^\top} W_{n'}^{m+1})$ . So

$$\overline{D}(K_{m+1}(Z^\top)) = \overline{KF}(K_{m+1}(Z^\top)).$$

This implies

$$D_{m+2}(Z^\top) = K_{m+2}(Z^\top) = KF_{m+2}(Z^\top).$$

Case 2. Suppose that  $m < \alpha$  is a limit ordinal and that the required statement holds for any  $\delta < m$ . Then

$$K_m(Z^\top) = \bigcup_{\delta < m} K_\delta(Z^\top) \cong \bigcup_{\delta < m} D_\delta(Z^\top) \cong \bigcup_{\delta < m} \sum_{\mathbb{N}^\top} W_{n'}^\delta \cong \sum_{\mathbb{N}^\top} \bigcup_{\delta < m} W_{n'}^\delta \cong \sum_{\mathbb{N}^\top} W_{n'}^m.$$

Now it is enough to show  $\overline{KF}(\sum_{\mathbb{N}^\top} W_{n'}^m) = \overline{D}(\sum_{\mathbb{N}^\top} W_{n'}^m)$ . Repeat the proof method of Case 1, we get that  $\overline{D}(K_m(Z^\top)) = \overline{KF}(K_m(Z^\top))$ . So

$$\overline{D}(K_m(Z^\top)) = \overline{K}(K_m(Z^\top)) = \overline{KF}(K_m(Z^\top)).$$

Hence

$$D_{m+1}(Z^\top) = K_{m+1}(Z^\top) = KF_{m+1}(Z^\top).$$

For (3), by (2), we get that

$$K_\alpha(Z^\top) = \bigcup_{\delta < \alpha} K_\delta(Z^\top) \cong \bigcup_{\delta < \alpha} KF_\delta(Z^\top) \cong KF_\alpha(Z^\top).$$

So  $K_\alpha(Z^\top)$  is well-filtered by (1) and hence  $K_\alpha(Z^\top)$  is  $k$ -well-filtered. Again by (1), we have that  $K_\delta(Z^\top)$  is not  $k$ -well-filtered for any ordinal  $\delta < \alpha$ . Therefore,  $\text{rank}_k(Z^\top) = \alpha$ .  $\square$

**Theorem 4.18** *For any ordinal  $\alpha$ , there exists a  $T_0$  space  $X$  and the following statements hold.*

- (1)  $X$  is  $(\alpha + 1)^d$ -special and  $(\alpha + 1)^{wf}$ -special.
- (2)  $\overline{D}(K_m(X)) = \overline{KF}(K_m(X))$  and  $D_{m+1}(X) = K_{m+1}(X) = KF_{m+1}(X)$  for  $0 \leq m < \alpha + 1$ .
- (3)  $X$  is  $(\alpha + 1)^k$ -special.

**Proof.** As usual, we use induction on  $\alpha$ . Base cases. If  $\alpha = 0$ , by Lemma 4.15,  $(N, \tau_\sigma)$  satisfies the required statements.

Inductive steps. There are two cases to consider:

Case (1). Suppose that the statements of the theorem hold for any ordinal  $\alpha$ . From Lemma 4.16, we have that the statements of the theorem hold for the ordinal  $\alpha + 1$ .

Case (2). Let  $\alpha$  be a limit ordinal. Assume for any ordinal  $\beta < \alpha$ , the statements of the theorem hold. Note that for every  $n \in N$ ,  $\bar{\alpha} + n < \alpha$ . So the statements of the theorem hold for  $\bar{\alpha} + n$ . By Lemma 4.17, it is straightforward to check that the statements of the theorem hold for  $\alpha$ .  $\square$

**Corollary 4.19** For any non-limit ordinal  $\alpha$ , there exists a  $T_0$  space  $X$  satisfying the following conditions.

- (1)  $X$  is  $\alpha^d$ -special and  $\alpha^{wf}$ -special.
- (2)  $\overline{D}(K_m(X)) = \overline{KF}(K_m(X))$  and  $D_{m+1}(X) = K_{m+1}(X) = KF_{m+1}(X)$  for  $0 \leq m < \alpha$ .
- (3)  $X$  is  $\alpha^k$ -special.

**Proof.** For any non-limit ordinal  $\alpha$ , there exists an ordinal  $\beta$  such that  $\alpha = \beta + 1$ . By Theorem 4.18, for ordinal  $\beta$ , there exists a  $T_0$  space  $X$  which satisfies the the required statements.  $\square$

**Theorem 4.20** For any ordinal  $\alpha$ , there exists an irreducible  $T_0$  space  $X$  such that the following statements hold.

- (1)  $\text{rank}_d(X) = \text{rank}_{wf}(X) = \alpha$ .
- (2)  $\overline{D}(K_m(X)) = \overline{KF}(K_m(X))$  and  $D_{m+1}(X) = K_{m+1}(X) = KF_{m+1}(X)$  for  $0 \leq m < \alpha$ .
- (3)  $\text{rank}_k(X) = \alpha$ .

**Proof.** Let  $\alpha$  be an ordinal. Now we consider two cases:

Case 1. If  $\alpha$  is not a limit ordinal, the statements of the theorem follow directly from Corollary 4.19.

Case 2. If  $\alpha$  is a limit ordinal, then  $\overline{\alpha} + n$  is a non-limit ordinal for any  $n \in \mathbb{N}$ . Hence, the statements of the theorem follow from Lemma 4.17 and Corollary 4.19.  $\square$

**Corollary 4.21** For any ordinal  $\alpha$ , there exists an irreducible  $T_0$  space  $X$  whose  $k$ -rank is equal to  $\alpha$ .

**Proof.** For any ordinal  $\alpha$ , it is enough to show that the  $k$ -rank of the space  $X$  required in Theorem 4.20 is  $\alpha$ .  $\square$

We can directly obtain the following corollaries.

**Corollary 4.22** (see also in [3]) For any ordinal  $\alpha$ , there exists an irreducible  $T_0$  space  $X$  whose  $d$ -rank is equal to  $\alpha$ .

**Corollary 4.23** (see also in [8]) For any ordinal  $\alpha$ , there exists an irreducible  $T_0$  space  $X$  whose  $wf$ -rank is equal to  $\alpha$ .

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