

Weakly Meet s_Z -continuity and δ_Z -continuity

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Abstract

Based on the concept of weakly meet s_Z -continuity put forward by Xu and Luo in [13], we further prove that if the subset system Z satisfies certain conditions, a poset is s_Z -continuous if and only if it is weakly meet s_Z -continuous and s_Z -quasicontinuous, which improves a related result given by Ruan and Xu in [10]. Meanwhile, we provide a characterization for the poset to be weakly meet s_Z -continuous, that is, a poset with a lower hereditary Z -Scott topology is weakly meet s_Z -continuous if and only if it is locally weakly meet s_Z -continuous. In addition, we introduce a monad on the new category \mathbf{POSET}_δ and characterize its *Eilenberg-Moore* algebras concretely.

Keywords: Weakly meet s_Z -continuous poset, δ_Z -continuous poset, Monad, Eilenberg-Moore algebras

1 Introduction

Recall that the concept of subset system on the category \mathbf{POSET} of posets was proposed by Wright et al. in [12]. It originally aimed at applying posets with Z -set structures to problems in computer science, particularly, to fixed point semantics for programming language. In addition, the set system includes many systems of sets which we are familiar with, such as directed sets, finite sets, connected sets and so on. Later, based on the suggestion given by Wright to study the generalized counterpart of continuous poset by replacing directed sets with Z -sets, in [2], Baranga defined a kind of generalized way-below relation based on the Z -sets whose supremum exists. Furthermore, the author gave some characterizations for the Z -algebraic posets. Besides, Ern e introduced the concept of s_2 -continuous posets by lending support to the cut operator of directed subsets instead of the existing sups, which is a pure order concept on posets, no longer depending on the dcpo. Recently, Zhang and Xu made use of the cut operator of directed sets again to define a new way-below relation between subsets, and then introduced s_2 -quasicontinuous posets (see [14]). Combining with the notion of subset system, Xu and Luo in [13] gave the definition of s_Z -quasicontinuous posets, and then, Ruan and Xu investigated its properties in [10] concretely and mainly made the conclusion: when the subset system satisfies some conditions, a poset is s_Z -continuous if and only if it is s_Z -quasicontinuous and meet s_Z -continuous.

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In this paper, we will first see that there is another characterization of s_Z -continuous posets. More precisely, in the case where the subset system Z satisfies certain conditions, a poset is s_Z -continuous if and only if it is s_Z -quasicontinuous and weakly meet s_Z -continuous, which is a result stronger than that given by Xu in [10], meanwhile, it reveals that the requirement of ' $\sigma^Z(P) = \sigma_Z(P)$ ' is unnecessary. Then we focus on the weakly meet s_Z -continuity of a poset, and show that a poset with a lower hereditary Z -Scott topology is weakly meet s_Z -continuous if and only if it is locally weakly meet s_Z -continuous. In order to investigate the Γ -faithful property, Ho and Zhao in [8] introduced a new beneath relation by the Scott closed subsets whose sups exist. Based on their work, we find there exists a monad on **DCPO**. Associating with the subset system, we will introduce a generalized beneath relation using the cuts of Z -Scott closed subsets, which is not necessary to consider whether the supremum exists. On this basis, we find a monad on the category **POSET** $_\delta$ and characterize its Eilenberg-Moore algebras concretely.

2 Preliminaries

Let P be a poset. For any $A \subseteq P$, $x \in P$, we write $\uparrow A = \{p \in P : p \geq a \text{ for some } a \in A\}$, $\downarrow A = \{p \in P : p \leq a \text{ for some } a \in A\}$. In particular, $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. A subset $A \subseteq P$ is an *upper set* (resp., a *lower set*) if $A = \uparrow A$ (resp., $A = \downarrow A$). Let A^u and A^l denote the sets of all upper and lower bounds of A , respectively. The cut operator δ is defined by $E^\delta = E^{ul}$ for all $E \subseteq P$. Obviously, if the supremum of E exists, then $x \in E^\delta$ iff $x \leq \sup E$. If $E \subseteq A$ for some subset A of P , let $E^\delta \upharpoonright_A = \{p \in A : p \leq m \text{ for all } m \in E^u \cap A\}$. In particular, we write $E^\delta \downarrow_m$ as $E^\delta \upharpoonright_m$. We denote by $F \subseteq_f P$ if F is a finite subset of poset P , and let $\mathbf{Fin}P = \{F : F \subseteq_f P\}$. A mapping $\mathbf{min} : \mathbf{Fin}P \rightarrow 2^P$ is defined by $\mathbf{min}(\uparrow F) = \{x \in F : x \text{ is a minimal element of } F\}$.

For any T_0 space X , the partial order \leq_X defined by $x \leq y$ iff x is contained in the closure of y is called the *specialization order*. The topology on the poset P generated by all principal filters $\uparrow x$ as a subbasis for the closed sets is called the *lower topology* and denoted by $\omega(P)$.

Let **POSET** denote the category of posets and monotone mappings. By [12], a *subset system* on **POSET** is a function Z which assigns to each poset, a set $Z(P)$ of subsets of P such that

- $\{x\} \in Z(P)$ for any $x \in P$, and
- if $f : P \rightarrow Q$ in **POSET** and $S \in Z(P)$, then $f(S) \in Z(Q)$.

P is called a *Z -complete poset* (*zcpo*, for short), if $\sup D$ exists for each $D \in Z(P)$. A closure system on the set X is a non-empty family \mathcal{E} of subsets of X which satisfies:

- $\bigcap_{i \in I} A_i \in \mathcal{E}$ for every nonempty family $\{A_i\}_{i \in I} \subseteq \mathcal{E}$, and
- $X \in \mathcal{E}$.

Definition 2.1 Let P be a poset and let $\sigma^Z(P) = \{U \subseteq P : \text{for all } S \in Z(P), S^\delta \cap U \neq \emptyset \Rightarrow S \cap U \neq \emptyset\}$. The topology generated by the subbasic open subsets $\sigma^Z(P)$ is called *Z -Scott topology* on P and denoted by $\sigma_Z(P)$.

Let $\Gamma^Z(P) = \{A \subseteq P : \text{for all } S \in Z(P), S \subseteq P \Rightarrow S^\delta \subseteq A\}$, obviously, $\Gamma^Z(P)$ is a subbasis for the closed subsets with respect to Z -Scott topology. We use $\Gamma_Z(P)$ to denote the set composed of all closed subsets regarding Z -Scott topology. Note that for any $U \in \sigma^Z(P)$ ($A \in \Gamma^Z(P)$), $U = \uparrow U$ ($A = \downarrow A$), so the definition above is the same as that given in [10]. Besides, the family $\Gamma^Z(P)$ and $\Gamma_Z(P)$ are closure systems on P , and the closure operators can be defined as follows: For any $M \subseteq P$, $cl_{\sigma_Z(P)}(M) = \bigcap \{A \in \sigma^Z(P) : M \subseteq A\}$, $cl_{\Gamma_Z(P)}(M) = \bigcap \{B \in \Gamma_Z(P) : M \subseteq B\}$.

Definition 2.2 ([10]) Let P be a poset and $x \in P$, $A, B \subseteq P$.

- (i) A is called *Z -way below* B , denoted by $A \ll_Z B$, if for any $S \in Z(P)$, $S^\delta \cap \uparrow B \neq \emptyset$ implies $S \cap \uparrow A \neq \emptyset$. $F \ll_Z \{x\}$ is shortly written as $F \ll_Z x$. Let $\omega_Z(x) = \{F \subseteq_f P : F \ll_Z x\}$, $\uparrow_Z A = \{x \in P : A \ll_Z x\}$, $\uparrow_Z A = \{p \in P : a \ll_Z p \text{ for some } a \in A\}$, $\downarrow_Z x = \{y \in P : y \ll_Z x\}$. Specifically, we write $\downarrow_Z^x y = \{m \in \downarrow x : m \ll_Z y \text{ in } \downarrow x\}$.

- (ii) P is called a weak s_Z -continuous poset, if for all $x \in P, x \in (\downarrow_Z x)^\delta$. In addition, if $\downarrow_Z x \in I_Z(P) = \{\downarrow S : S \in Z(P)\}$, then P is called s_Z -continuous.
- (iii) P is called an s_Z -quasicontinuous poset, if for all $p \in P, \{\uparrow F : F \in \omega_Z(P)\} \in Z(\mathbf{Fin}P)$ and $\uparrow p = \bigcap \{\uparrow F : F \in \omega_Z(P)\}$.

Definition 2.3 ([15]) A subset system Z is hereditary if for any order embedding $f : P \rightarrow Q$ (that is, for any $x, y \in P, f(x) \leq f(y) \Leftrightarrow x \leq y$), $D \subseteq P, D \in Z(P)$ if and only if $f(D) \in Z(Q)$.

Definition 2.4 ([10]) Let Z be a subset system.

- (i) Z is called *union complete*, if for any poset $P, \mathcal{S} \in Z(Z(P))$, we have $\bigcup \mathcal{S} \in Z(P)$.
- (ii) Z is said to have the *finite family union property*, if for any poset $P, \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n\} \subseteq_f Z(\mathbf{Fin}P)$, we have $\{\bigcup_{i=1}^n A_i : A_i \in \mathcal{S}_i, i = 1, 2, \dots, n\} \in Z(\mathbf{Fin}P)$.
- (iii) Z is said to have the property M , if for any poset $P, \uparrow F \in \mathbf{Fin}P$, we have $\downarrow_{\mathbf{Fin}P} \uparrow F = \{\uparrow G \in \mathbf{Fin}P : \uparrow F \subseteq \uparrow G\} \in Z(\mathbf{Fin}P)$.

Definition 2.5 ([10]) A subset system Z is said to have the *Rudin property*, if for any poset $P, E = \uparrow E \subseteq P, \mathcal{G} \in Z(\mathbf{Fin}P), \emptyset \notin \mathcal{G}$, and $\bigcap \mathcal{G} \subseteq E$. Then there exists $K \subseteq \bigcup \{\mathbf{min}(G) : G \in \mathcal{G}\}$ such that

- (i) for any $G \in \mathcal{G}, K \cap \mathbf{min}(G) \neq \emptyset$,
- (ii) $K \in Z(P)$,
- (iii) $\bigcap \{\uparrow k : k \in K\} \subseteq E$, and
- (iv) for any $G, H \in \mathcal{G}, G \subseteq H$ implies $K \cap \mathbf{min}(G) \subseteq \uparrow(K \cap \mathbf{min}(H))$.

Z is called a *Rudin subset system*, if Z is union-complete and possesses the Rudin property.

3 Weakly meet s_Z -continuous posets

Definition 3.1 ([10]) P is called weakly meet s_Z -continuous if for all $x \in P$ and all $D \in Z(P)$ with $x \in D^\delta$, we have $x \in cl_{\sigma_Z(P)}(\downarrow x \cap \downarrow D)$; P is called meet s_Z -continuous if for all $x \in P$ and all $D \in Z(P)$ with $x \in D^\delta$, we have $x \in cl_{\sigma_Z(P)}(\downarrow x \cap \downarrow D)$.

Lemma 3.2 Let P be a poset. The following conditions are equivalent:

- (1) P is weakly meet s_Z -continuous.
- (2) For any $x \in P$ and any $U \in \sigma^Z(P), \uparrow(\downarrow x \cap U) \in \sigma^Z(P)$.

Proof. (1) \Rightarrow (2) Assume that $D \in Z(P)$, and $D^\delta \cap \uparrow(\downarrow x \cap U) \neq \emptyset$. Then there exists an $m \in D^\delta$ with $m \in U$ and $m \leq x$. Since P is weakly meet s_Z -continuous, we have $m \in cl_{\sigma_Z(P)}(\downarrow m \cap \downarrow D)$, which implies that $\downarrow D \cap \downarrow m \cap U \neq \emptyset$. Thus $\downarrow D \cap \downarrow x \cap U \neq \emptyset$ by $m \leq x$. So $D \cap \uparrow(\downarrow x \cap U) \neq \emptyset$ and $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$ holds.

(2) \Rightarrow (1) For any $x \in P, D \in Z(P)$, if $x \in D^\delta$ and there is a $U \in \sigma^Z(P)$ such that $x \in U$, then by (2), $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$. Since $x \in D^\delta \cap \uparrow(\downarrow x \cap U) \neq \emptyset$, we have $D \cap \uparrow(\downarrow x \cap U) \neq \emptyset$, this means $\downarrow x \cap U \cap \downarrow D \neq \emptyset$. So $x \in cl_{\sigma_Z(P)}(\downarrow x \cap \downarrow D)$. \square

Lemma 3.3 Let P be a Z -complete semilattice. The following conditions are equivalent:

- (1) P is weakly meet s_Z -continuous;
- (2) For any $x \in P, D \in Z(P), x \wedge \vee D = \vee \{x \wedge d : d \in D\}$.

Proof. (1) \Rightarrow (2) We first claim that $y = \vee(\downarrow y \cap \downarrow D)$ if $y \in D^\delta$. It is obvious that y is an upper bound of $\downarrow y \cap \downarrow D$. Suppose z is also an upper bound of $\downarrow y \cap \downarrow D$ and $y \not\leq z$, that is, $y \in P \setminus \downarrow z$. Since $y \in cl_{\sigma_Z(P)}(\downarrow y \cap \downarrow D)$ by (1) and $P \setminus \downarrow z \in \sigma^Z(P)$, we have $(P \setminus \downarrow z) \cap \downarrow y \cap \downarrow D \neq \emptyset$. But this contradicts the fact that $\downarrow y \cap \downarrow D \subseteq \downarrow z$. Thus $y = \vee(\downarrow y \cap \downarrow D)$. Now let $y_0 = x \wedge \vee D$, then $y_0 \in D^\delta$, which implies

$y_0 = \vee(\downarrow y_0 \cap \downarrow D)$. Since $\downarrow y_0 \cap \downarrow D = \downarrow\{x \wedge d : d \in D\}$, we have $y_0 = \vee\{x \wedge d : d \in D\}$, that is, $x \wedge \vee D = \vee\{x \wedge d : d \in D\}$.

(2) \Rightarrow (1) For any $x \in P$, $U \in \sigma^Z(P)$, we need to prove $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$. Assume $D \in Z(P)$ with $D^\delta \cap \uparrow(\downarrow x \cap U) \neq \emptyset$. Then there exists an $m \in U$, $m \leq x$ and $m \in D^\delta$. Thus $m \leq \vee D$ and $m = m \wedge \vee D = \vee\{m \wedge d : d \in D\} \in U$ by (2). Now for m , we define a monotone mapping $\varphi : P \rightarrow P$ by $\varphi(p) = m \wedge p$. Then $\varphi(D) = \{m \wedge d : d \in D\} \in Z(P)$. Hence, $m \wedge d_0 \in U$ for some $d_0 \in D$ as $U \in \sigma^Z(P)$, which implies that $D \cap \uparrow(\downarrow x \cap U) \neq \emptyset$, that is, $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$. So P is weakly meet s_Z -continuous by Lemma 3.2. \square

Proposition 3.4 *Let P be a poset. The following conditions are equivalent:*

- (1) P is weakly meet s_Z -continuous;
- (2) $\Gamma^Z(P)$ is weakly meet s_Z -continuous.

Proof. (1) \Rightarrow (2) By Lemma 3.3, we only need to prove that for any $A \in \Gamma^Z(P)$, $\mathcal{D} \in Z(\Gamma^Z(P))$, $A \wedge (\vee \mathcal{D}) = \vee\{A \wedge D : D \in \mathcal{D}\}$, that is, $A \cap cl_{\sigma^Z(P)}(\bigcup \mathcal{D}) = cl_{\sigma^Z(P)}(\bigcup\{A \cap D : D \in \mathcal{D}\})$. Assume $x \in A \cap cl_{\sigma^Z(P)}(\bigcup \mathcal{D})$ and $U \in \sigma^Z(P)$ with $x \in U$. Then we have $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$ by Lemma 3.2. As $x \in cl_{\sigma^Z(P)}(\bigcup \mathcal{D})$, $\uparrow(\downarrow x \cap U) \cap D_0 \neq \emptyset$ for some $D_0 \in \mathcal{D}$, this means $A \cap U \cap D_0 \neq \emptyset$ since $x \in A$ and A is a lower set. Moreover, $(\bigcup\{A \cap D : D \in \mathcal{D}\}) \cap U \neq \emptyset$. So $x \in cl_{\sigma^Z(P)}(\bigcup\{A \cap D : D \in \mathcal{D}\})$, and $A \cap cl_{\sigma^Z(P)}(\bigcup \mathcal{D}) \subseteq cl_{\sigma^Z(P)}(\bigcup\{A \cap D : D \in \mathcal{D}\})$ holds. Obviously, the conversely inclusion holds.

(2) \Rightarrow (1) It is sufficient to prove that $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$ for any $x \in P$, $U \in \sigma^Z(P)$. Let $D \in Z(P)$ with $D^\delta \cap \uparrow(\downarrow x \cap U) \neq \emptyset$. Then there exists an $m \in \downarrow x \cap U$ such that $m \in D^\delta$, which implies $\downarrow m \in \{\downarrow d : d \in D\}^\delta$. In addition, we know $\{\downarrow d : d \in D\} \in Z(\Gamma^Z(P))$ since the mapping $\psi : P \rightarrow \Gamma^Z(P)$ defined by $\psi(p) = \downarrow p$ is monotone. As $\Gamma^Z(P)$ is weakly meet s_Z -continuous, we have $\downarrow m \in cl_{\sigma^Z(\Gamma^Z(P))}(\downarrow\{\downarrow m\} \cap \downarrow\{\downarrow d : d \in D\})$. It is easy to verify that $\diamond U = \{A \in \Gamma^Z(P) : A \cap U \neq \emptyset\} \in \sigma^Z(\Gamma^Z(P))$ and $\downarrow m \in \diamond U$. So $\diamond U \cap \downarrow\{\downarrow m\} \cap \downarrow\{\downarrow d : d \in D\} \neq \emptyset$, that is, $C \in \Gamma^Z(P)$ belongs to this intersection. Moreover, there exists an element $c \in C \cap U$ satisfying $c \leq m \leq x$ and $c \leq d_0$ for some $d_0 \in D$, this means $D \cap \uparrow(\downarrow x \cap U) \neq \emptyset$. Hence, $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$, and P is weakly meet s_Z -continuous. \square

Lemma 3.5 *Let P be a weakly meet s_Z -continuous poset. If F is a finite subset of P , then $int_{\sigma^Z(P)}(\uparrow F) \subseteq \cup\{\uparrow_Z x : x \in F\}$.*

Proof. Suppose $F = \{x_1, x_2, \dots, x_n\}$ and there exists an element $a \in int_{\sigma^Z(P)}(\uparrow F)$, but $a \notin \cup\{\uparrow x_i : i = 1, 2, \dots, n\}$. Then $x_i \not\ll_Z a$ for any $x_i \in F$, that is, there exists $D_i \in Z(P)$ such that $a \in D_i^\delta$, but $x_i \notin \downarrow D_i$, for $i = 1, 2, \dots, n$. For $D_1 \in Z(P)$ with $a \in D_1^\delta$, $a \in cl_{\sigma^Z(P)}(\downarrow a \cap \downarrow D_1)$ by weakly meet s_Z -continuity. Then $int_{\sigma^Z(P)}(\uparrow F) \cap \downarrow a \cap \downarrow D_1 \neq \emptyset$, which implies that there is a $y_1 \in int_{\sigma^Z(P)}(\uparrow F) \cap \downarrow a \cap \downarrow D_1$. By $y_1 \leq a$ and $a \in D_2^\delta$, we have $y_1 \in D_2^\delta$. Similarly, we get that $y_1 \in cl_{\sigma^Z(P)}(\downarrow y_1 \cap \downarrow D_2)$ and $int_{\sigma^Z(P)}(\uparrow F) \cap \downarrow y_1 \cap \downarrow D_2 \neq \emptyset$. So there is a $y_2 \in int_{\sigma^Z(P)}(\uparrow F) \cap \downarrow y_1 \cap \downarrow D_2$. By induction, we find $y_n \in int_{\sigma^Z(P)}(\uparrow F) \cap \downarrow y_{n-1} \cap \downarrow D_n$, where $y_0 = a$, clearly, $y_n \in \bigcap_{i=1}^n \downarrow D_i$. Since $y_n \in int_{\sigma^Z(P)}(\uparrow F) \subseteq \uparrow F$, $y_n \geq x_{i_0}$ for some $i_0 \in \{1, 2, \dots, n\}$, this implies $x_{i_0} \in \downarrow D_{i_0}$, which contradicts that $x_{i_0} \notin \downarrow D_{i_0}$. Hence, $int_{\sigma^Z(P)}(\uparrow F) \subseteq \cup\{\uparrow_Z x : x \in F\}$. \square

Lemma 3.6 ([10]) *Let Z be a Rudin subset system which has the finite family union property and P an s_Z -quasicontinuous poset. Then the following statements hold.*

- (1) For any finite set F in P , $\uparrow_Z F \in \sigma^Z(P)$.
- (2) If $U \subseteq P$, then $U \in \sigma^Z(P)$ if and only if for any $x \in U$, there exists $F \subseteq_f P$ such that $x \in \uparrow_Z F \subseteq \uparrow F \subseteq U$.

Lemma 3.7 *Let Z be a Rudin subset system which has the finite family union property. If P is weakly meet s_Z -continuous and s_Z -quasicontinuous, then for any finite subset F of P , we have*

$$\uparrow_Z F = \uparrow_Z F.$$

Proof. By Lemma 3.5 and Lemma 3.6, obviously, $\uparrow_Z F \subseteq \uparrow_Z F$. And the reverse containment is easy to verify, so we omit the proof. \square

Proposition 3.8 ([13]) *Let Z be a Rudin subset system which possesses M property. If P is an s_Z -continuous poset, then P is s_Z -quasicontinuous, and for any $p \in P$, $\omega_Z(p) = \{F \subseteq_f P : \exists y \ll_Z p, \text{ such that } y \in \uparrow F\}$.*

Proposition 3.9 ([10]) *Let P be an s_Z -continuous poset. Then P is weakly meet s_Z -continuous.*

Proposition 3.10 *Let P be a weakly meet s_Z -continuous poset. If for any $x, y \in P$, $x \not\leq y$, there are $U \in \sigma^Z(P)$, $V \in \omega(P)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$, then P is weak s_Z -continuous.*

Proof. It suffices to prove that $x \in (\downarrow_Z x)^\delta$ for any $x \in P$. Suppose that there is a $y \in (\downarrow_Z x)^u$ but $x \not\leq y$. Then there are $U \in \sigma^Z(P)$, $V = P \setminus \uparrow F \in \omega(P)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$, so $U \subseteq \uparrow F$. Since $\uparrow(\downarrow x \cap U) \in \sigma^Z(P)$ by Lemma 3.2 and $x \in \uparrow(\downarrow x \cap U) \subseteq \uparrow F$, we have $x \in \text{int}_{\sigma^Z(P)}(\uparrow F) \subseteq \uparrow_Z F$. Thus there is an $m \in F$ such that $m \in \downarrow_Z x$. It follows that $m \leq y$, then $y \in \uparrow F$. But this contradicts that $y \in V$. \square

Theorem 3.11 *Let P be a poset and Z a Rudin subset system which possesses the finite family union property and M property. If $\downarrow_Z x \in I_Z(P)$ for each $x \in P$, then the following conditions are equivalent:*

- (1) P is s_Z -continuous;
- (2) P is weakly meet s_Z -continuous and s_Z -quasicontinuous;
- (3) P is weakly meet s_Z -continuous, and for any $x \not\leq y$ in P , there are $U \in \sigma^Z(P)$, $V \in \omega(P)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Proof. (1) \Rightarrow (2) Straightforward by Proposition 3.8 and Proposition 3.9.

(2) \Rightarrow (3) For any $x \not\leq y$, that is, $y \notin \uparrow x$, there is an $F \in \omega_Z(x)$ such that $y \notin \uparrow F$ by (2). So we get that there are $\uparrow_Z F \in \sigma^Z(P)$, $P \setminus \uparrow F \in \omega(P)$ containing x and y , respectively, and $\uparrow_Z F \cap P \setminus \uparrow F = \emptyset$.

(3) \Rightarrow (1) By Proposition 3.10 and $\downarrow_Z x \in I_Z(P)$, we know P is s_Z -continuous. \square

4 Posets with lower hereditary Z -Scott topology

Definition 4.1 Let P be a poset. The Z -Scott topology on P is called *lower hereditary* if for each closed subbasis A of P , the Z -Scott topology of poset A is precisely generated by the subbasic closed subsets of the form $B \cap A$, where $B \in \Gamma^Z(P)$, that is, $\Gamma^Z(A) = \{B \cap A : B \in \Gamma^Z(P)\}$.

Definition 4.2 Let P, Q be two posets. A mapping $f : P \rightarrow Q$ is called σ^Z -continuous if for any $A \in \Gamma^Z(Q)$, $f^{-1}(A) \in \Gamma^Z(P)$.

It is obvious that f is monotone if f is σ^Z -continuous.

Lemma 4.3 *Let P and Q be two posets and $f : P \rightarrow Q$. Consider the following three conditions:*

- (1) f is σ^Z -continuous.
- (2) For any $D \in Z(P)$, $f(D^\delta) \subseteq f(D)^\delta$.
- (3) $f(\text{cl}_{\sigma^Z(P)}(A)) \subseteq \text{cl}_{\sigma^Z(Q)}(f(A))$ for each $A \subseteq P$.

Then (1) \Leftrightarrow (2) \Rightarrow (3).

Proof. Straightforward. \square

Lemma 4.4 *Let P be a poset and Z a subset hereditary subset system. Consider the following conditions:*

- (1) The Z -Scott topology on P is lower hereditary.
- (2) The inclusion map $i : \downarrow x \rightarrow P$ is σ^Z -continuous for any $x \in P$.
- (3) For any $x \in P$ and $D \in Z(\downarrow x)$, $D^\delta \upharpoonright_x = D^\delta$.

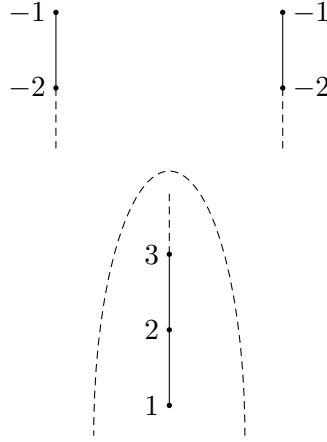


Fig. 1.

(4) For any $A \in \Gamma^Z(P)$ and $D \in Z(A)$, $D^\delta|_A = D^\delta$.

(5) For any $D \in Z(P)$, D^u is filtered.

Then (5) \Rightarrow (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4).

Proof. It is easy to verify that (1) \Rightarrow (2) \Rightarrow (3).

(3) \Rightarrow (4) : It is clear that $D^\delta \subseteq D^\delta|_A$. Assume $m \in D^u|_A$. Then $D \subseteq \downarrow m$ and $m \in A$. Since Z is subset hereditary, $D \in Z(\downarrow m)$. Thus we have $D^\delta|_m = D^\delta$ by (3). Now we only need to prove that $D^\delta|_A \subseteq D^\delta|_m$. Assume $a \in D^\delta|_A$, $b \in D^u|_m$. Then $b \leq m$ and $b \in A$ as $m \in A$, which implies that $b \in D^u|_A$, so $a \leq b$. Hence, $D^\delta|_A \subseteq D^\delta|_m$.

(4) \Rightarrow (1) : We want to prove that $\Gamma^Z(A) = \{A \cap C : C \in \Gamma^Z(P)\}$ for any $A \in \Gamma^Z(P)$. For each $B \in \Gamma^Z(A)$, let $D \in Z(P)$ and $D \subseteq B$. Then $D \in Z(A)$ because Z is subset hereditary. It follows that $D^\delta|_A \subseteq B$, which means $D^\delta \subseteq B$ since $D^\delta|_A = D^\delta$. Thus $B \in \Gamma^Z(P)$ and $\Gamma^Z(A) \subseteq \{A \cap C : C \in \Gamma^Z(P)\}$. Conversely, for any $C \in \Gamma^Z(P)$, let $D \in Z(A)$ with $D \subseteq A \cap C$. Then $D \in Z(P)$ and $D^\delta \subseteq A \cap C$ since $A \cap C \in \Gamma^Z(P)$. This implies that $D^\delta|_A \subseteq A \cap C$. So $A \cap C \in \Gamma^Z(A)$, and hence, $\Gamma^Z(A) = \{A \cap C : C \in \Gamma^Z(P)\}$ holds.

(5) \Rightarrow (3) : Clearly, $D^\delta \subseteq D^\delta|_x$. Conversely, assume $m \in D^\delta|_x$, $n \in D^u$. Then $x, n \in D^u$. Since D^u is filtered, there is a $p \in D^u$ such that $p \leq x, n$. This implies $p \in D^u|_x$, so $m \leq p$. It follows that $m \leq n$ by $p \leq n$. Thus $D^\delta|_x \subseteq D^\delta$. \square

Example 4.5 The condition (5) in the above lemma is not equivalent to others. Let \mathbb{N} be the set of natural numbers and $P = \mathbb{N} \cup (\mathbb{N}^\partial \dot{\cup} \mathbb{N}^\partial)$ with the partial order defined by $x \leq y$ iff $x \leq y$ in \mathbb{N} or $x \leq y$ in \mathbb{N}^∂ or $x \in \mathbb{N}$ and $y \in \mathbb{N}^\partial \dot{\cup} \mathbb{N}^\partial$ (see Fig. 1 for a better understanding). One can easily see that for any $-n \in \mathbb{N}^\partial \dot{\cup} \mathbb{N}^\partial$ and $D = \mathbb{N} \in \mathcal{D}(\downarrow\{-n\})$, $D^\delta|_{-n} = D^\delta$, but D^u is not filtered.

Corollary 4.6 Every $zcpo$ P has a lower hereditary Z -Scott topology.

Proof. Since $\sup D$ exists for any $D \in Z(P)$, D^u is filtered. \square

Definition 4.7 A poset P is called locally weakly meet s_Z -continuous if $\downarrow x$ as a subposet of P is weakly meet s_Z -continuous for each $x \in P$.

Lemma 4.8 Let P be a poset with a lower hereditary Z -Scott topology and $A \in \Gamma^Z(P)$. Then for any $E \subseteq A$, we have $cl_{\sigma^Z(P)}(E) = cl_{\sigma^Z(A)}(E)$.

Proof. Straightforward. \square

Theorem 4.9 Let P be a poset with a lower hereditary Z -Scott topology and Z be subset hereditary. Then P is weakly meet s_Z -continuous if and only if P is locally weakly meet s_Z -continuous.

Proof. (\Rightarrow): For any $x \in P$, let $D \in Z(\downarrow x)$, $y \in D^\delta \upharpoonright_x$. Then $D \in Z(P)$. Since the Z -Scott topology of P is lower hereditary, by Lemma 4.4, we have $D^\delta \upharpoonright_x = D^\delta$. Thus $y \in D^\delta$. It follows that $y \in cl_{\sigma_Z(P)}(\downarrow y \cap \downarrow D)$ by the weakly meet s_Z -continuity of P . Therefore, $y \in cl_{\sigma_Z(\downarrow x)}(\downarrow y \cap \downarrow D)$ by Lemma 4.8.

(\Leftarrow): Suppose $D \in Z(P)$, $y \in D^\delta$. For any $m \in D^u$, we have $D \subseteq \downarrow m$ and $D \in Z(\downarrow m)$ as Z is subset hereditary. Since $D^\delta = D^\delta \upharpoonright_m$, we have $y \in D^\delta \upharpoonright_m$, which implies that $y \in cl_{\sigma_Z(\downarrow m)}(\downarrow y \cap \downarrow D)$. So $y \in cl_{\sigma_Z(P)}(\downarrow y \cap \downarrow D)$ by Lemma 4.8 again. \square

Proposition 4.10 *Let P be a poset with a lower hereditary Z -Scott topology. If P is weak s_Z -continuous and for any $x \in P$, $y \in \downarrow x$, $\downarrow_Z^x y \in Z(\downarrow x)$, then $\downarrow x$ is s_Z -continuous.*

Proof. We need to prove that $y \in (\downarrow_Z^x y)^\delta \upharpoonright_x$. Since P is weak s_Z -continuous, $y \in (\downarrow_Z y)^\delta$. Assume $m \ll_Z y$, $D \in Z(\downarrow x)$ with $y \in D^\delta \upharpoonright_x$. Then $y \in D^\delta$ since the Z -Scott topology on P is lower hereditary. So $m \in \downarrow D$ by $m \ll_Z y$, which implies that $m \ll_Z y$ in $\downarrow x$. Therefore, $\downarrow_Z y \subseteq \downarrow_Z^x y$. Hence, $y \in (\downarrow_Z y)^\delta \subseteq (\downarrow_Z^x y)^\delta = (\downarrow_Z^x y)^\delta \upharpoonright_x$, where the last equality holds as $\downarrow_Z^x y \in Z(\downarrow x)$. Moreover, $\downarrow_Z^x y \in Z(\downarrow x)$ implies $\downarrow_Z^x y \in I_Z(\downarrow x)$, so $\downarrow x$ is s_Z -continuous. \square

Proposition 4.11 *Let P be a poset with a lower hereditary Z -Scott topology and Z be subset hereditary. If for any $x \in P$, $\downarrow x$ is s_Z -continuous and $\downarrow_Z x \in Z(P)$, then P is s_Z -continuous.*

Proof. We only need to prove that $x \in (\downarrow_Z x)^\delta$. By assumption, $\downarrow x$ is s_Z -continuous, we have $x \in (\downarrow_Z x)^\delta \upharpoonright_x$. Now we show that $\downarrow_Z^x x \subseteq \downarrow_Z x$. Let $m \ll_Z x$ in $\downarrow x$ and $D \in Z(P)$ with $x \in D^\delta$. We can find that $D \subseteq \downarrow y$ and $D \in Z(\downarrow y)$ for each $y \in D^u$. Claim that $\downarrow_Z^x x \subseteq \downarrow_Z^y x$. Assume $a \in \downarrow_Z^x x$. Since $\downarrow y$ is s_Z -continuous and $x \in \downarrow y$, we have $\downarrow_Z^y x \in Z(\downarrow y)$ and $x \in (\downarrow_Z^y x)^\delta \upharpoonright_y$. It follows that $x \in (\downarrow_Z^y x)^\delta \upharpoonright_x$ as $\downarrow x \subseteq \downarrow y$. Moreover, $\downarrow_Z^y x \in Z(\downarrow x)$ as $\downarrow_Z^y x \subseteq \downarrow x$ and Z is subset hereditary. This implies that $a \in \downarrow_Z^y x$. So $\downarrow_Z^x x \subseteq \downarrow_Z^y x$ holds. Thus $m \ll_Z x$ in $\downarrow y$. As $x \in D^\delta$ implies that $x \in D^\delta \upharpoonright_y$, we have $m \in \downarrow D$. Hence, $m \ll_Z x$. Then $\downarrow_Z^x x \subseteq \downarrow_Z x$. It is self-evident that $x \in (\downarrow_Z^x x)^\delta \upharpoonright_x \subseteq (\downarrow_Z x)^\delta \upharpoonright_x = (\downarrow_Z x)^\delta$, where the last equality holds as $\downarrow_Z x \in Z(\downarrow x)$. So P is s_Z -continuous. \square

Theorem 4.12 *Let P be a poset with a lower hereditary Z -Scott topology and Z be subset hereditary. Then the following conditions are equivalent:*

- (1) P is s_Z -continuous and $\downarrow_Z^x y \in Z(\downarrow x)$ for any $x \in P$ and $y \in \downarrow x$;
- (2) $\downarrow x$ is s_Z -continuous and $\downarrow_Z x \in Z(P)$ for any $x \in P$.

Proof. Straightforward by Proposition 4.10 and 4.11. \square

5 A monad on \mathbf{POSET}_δ

In this part, \mathbf{POSET}_δ denotes the category whose objects are all posets and morphisms are σ^Z -continuous mappings. We will give a monad on \mathbf{POSET}_δ and characterize its Eilenberg-Moore algebras.

Definition 5.1 Let P be a poset and $x, y \in P$.

- (1) x is called Z -beneath y , denoted by $x \prec_Z y$, if for any $A \in \Gamma^Z(P)$ with $y \in A^\delta$, $x \in A$.
- (2) P is said to be δ_Z -continuous if for all $a \in P$, $a \in \{m \in P : m \prec_Z a\}^\delta$.

Notice that the set $\{m \in P : m \prec_Z a\} \in \Gamma^Z(P)$ automatically. There are some common properties about the relation \prec_Z being similar to the \ll .

Proposition 5.2 *Let P be a poset and $x, y, m, n \in P$.*

- (1) $x \prec_Z y$ implies $x \leq y$;
- (2) $m \leq x \prec_Z y \leq n$ implies $m \prec_Z n$;
- (3) if P has a bottom 0, then $0 \prec_Z x$ always holds.

Proposition 5.3 *Let P be a poset and $\mathcal{C} \in \Gamma^Z(\Gamma^Z(P))$. Then the supremum of \mathcal{C} in $\Gamma^Z(P)$ exists and is exactly $\bigcup \mathcal{C}$.*

Proof. Clearly, it is enough to show that $\bigcup \mathcal{C} \in \Gamma^Z(P)$. For any $D \in Z(P)$ with $D \subseteq \bigcup \mathcal{C}$, there is $C_d \in \mathcal{C}$ for each $d \in D$ such that $d \in C_d$. Then we have $\downarrow d \subseteq C_d$ and $\{\downarrow d : d \in D\} \subseteq \mathcal{C}$ as \mathcal{C} is a lower set. Since the monotonicity of the mapping $f : P \rightarrow \Gamma^Z(P)$ defined by $f(p) = \downarrow p$ implies that $\{\downarrow d : d \in D\} \in Z(\Gamma^Z(P))$, $\{\downarrow d : d \in D\}^\delta \subseteq \mathcal{C}$. Now consider each $a \in D^\delta$, we have $\downarrow a \in \{\downarrow d : d \in D\}^\delta$, which means $\downarrow a \in \mathcal{C}$. Thus $a \in \bigcup \mathcal{C}$, and $D^\delta \subseteq \bigcup \mathcal{C}$ holds. \square

Definition 5.4 Let P be a poset.

- (1) An element x of P is called *Z-compact* if $x \prec_Z x$. We use $k_Z(P)$ to denote the set of all *Z-compact* elements of P .
- (2) P is called *δ_Z -prealgebraic* if for each $x \in P$, $x \in \{y \in k_Z(P) : y \leq x\}^\delta$.

Notably, we call a *δ_Z -prealgebraic* complete lattice a *δ_Z -prealgebraic* lattice for short. Obviously, $\Gamma^Z(P)$ is a *δ_Z -prealgebraic* lattice for any poset P .

Lemma 5.5 *Let (g, d) be a Galois connection between two posets S and T , where $g : S \rightarrow T$, $d : T \rightarrow S$. Then d preserves cuts of any subset of T , that is, $d(A^\delta) \subseteq d(A)^\delta$ for any $A \subseteq T$.*

Proof. It suffices to show that $d(x) \in d(A)^\delta$ for any $x \in A^\delta$. Let y be an upper bound of $d(A)$. Then for each $a \in A$, we have $d(a) \leq y$, and so $a \leq g(y)$. It follows that $A \subseteq \downarrow g(y)$. Thus $A^\delta \subseteq \downarrow g(y)$, which implies $x \leq g(y)$, so $d(x) \leq y$. Hence, $d(x) \in d(A)^\delta$. \square

Lemma 5.6 *Let (g, d) be a Galois connection between two posets S and T , where $g : S \rightarrow T$, $d : T \rightarrow S$. Then for any $C \in \Gamma^Z(S)$, $\downarrow g(C) \in \Gamma^Z(T)$.*

Proof. Let E be a *Z-set* of T with $E \subseteq \downarrow g(C)$. Then for any $e \in E$, there is a $c_e \in C$ such that $e \leq g(c_e)$, this means $d(e) \leq c_e$. Thus $d(E) \subseteq C$ and $d(E)^\delta \subseteq C$ since $C \in \Gamma^Z(S)$. The conclusion of Lemma 5.5 indicates that $d(E^\delta) \subseteq C$. Therefore, $E^\delta \subseteq d^{-1}(C) = \downarrow g(C)$. \square

Lemma 5.7 *Let (g, d) be a Galois connection between two posets S and T , where $g : S \rightarrow T$, $d : T \rightarrow S$. Consider the following two conditions:*

- (1) *For any $A \in \Gamma^Z(S)$, $g(A^\delta) \subseteq g(A)^\delta$.*
- (2) *d preserves the relation \prec_Z .*

Then (1) \Rightarrow (2); if T is δ_Z -continuous, then (2) \Rightarrow (1).

Proof. (1) \Rightarrow (2): We need to show that $d(x) \prec_Z d(y)$ for any $x \prec_Z y$ in T . Let $A \in \Gamma^Z(S)$ with $d(y) \in A^\delta$. Then $y \leq g(m)$ for some $m \in A^\delta$. By the condition (1), we have $g(m) \in g(A)^\delta$ and hence, $y \in g(A)^\delta = (\downarrow g(A))^\delta$. Lemma 5.6 indicates that $\downarrow g(A) \in \Gamma^Z(T)$, then $x \in \downarrow g(A)$ as $x \prec_Z y$. Thus there is an $a \in A$ such that $x \leq g(a)$, which implies $d(x) \leq a$. It follows that $d(x) \in A$. Therefore, d preserves the relation \prec_Z .

(2) \Rightarrow (1): By the δ_Z -continuity of T , we know $g(x) \in \{y \in T : y \prec_Z g(x)\}^\delta$ for each $x \in A^\delta$. Thus in order to show $g(A^\delta) \subseteq g(A)^\delta$ for any $A \in \Gamma^Z(S)$, it suffices to prove that for each $x \in A^\delta$, $\{y \in T : y \prec_Z g(x)\}^\delta \subseteq g(A)^\delta$. For each $y \prec_Z g(x)$, since $d(y) \prec_Z d(g(x)) \leq x$, we have $d(y) \prec_Z x$. Then $d(y) \in A$ because $x \in A^\delta$ and $A \in \Gamma^Z(S)$, which implies $y \in \downarrow g(A)$. Thus $\{y \in T : y \prec_Z g(x)\}^\delta \subseteq (\downarrow g(A))^\delta = g(A)^\delta$. \square

Lemma 5.8 *If L is a *zcpo*, then $k_Z(L)$ is also a *zcpo*.*

Proof. We just need to prove that $\sup D \in k_Z(L)$ for any $D \in Z(k_Z(L))$. Let $A \in \Gamma^Z(L)$ with $\sup D \in A^\delta$. Then $D \subseteq A^\delta$, and so $D \subseteq A$ by $D \subseteq k_Z(L)$. Thus $\downarrow \sup D = D^\delta \subseteq A$, this means $\sup D \in A$. Hence, $\sup D \in k_Z(L)$ and $\sup_{k_Z(L)} D = \sup D \in k_Z(L)$. It follows that $k_Z(L)$ is a *zcpo*. \square

The above lemma ensures that $D^\delta \downarrow_{k_Z(L)} = D^\delta \cap k_Z(L)$ hold. There is an example illustrating that $D^\delta \downarrow_{k_Z(L)} = D^\delta \cap k_Z(L)$ doesn't hold if L is not a *zcpo*.

Example 5.9 Let P be the poset consist of all natural numbers \mathbb{N} and $\{a, b, c, d, \top\}$. \top is the greatest element of P and $\{a, b, c, d\} \subseteq \mathbb{N}^u$, $c \in \{a, b\}^u$. Now consider $Z = D$, where $D(P)$ is the family of all directed subsets. It is easy to verify that $k_D(P) = \mathbb{N} \cup \{d\}$. For $\mathbb{N} \in D(P)$, $\mathbb{N}^\delta \downarrow_{k_D(P)} = \mathbb{N} \cup \{d\}$, however, $\mathbb{N}^\delta \cap k_D(P) = \mathbb{N}$ since $\mathbb{N}^\delta = \mathbb{N}$.

We denote by $\delta_Z\mathbf{PALG}$ the category which has all δ_Z -prealgebraic lattices as objects and maps that have an upper adjoint and preserve the relation \prec_Z as morphisms. Next, we will investigate the relation between the categories \mathbf{POSET}_δ and $\delta_Z\mathbf{PALG}$.

Theorem 5.10 Let K_Z and Γ^Z be two functors between $\delta_Z\mathbf{PALG}$ and \mathbf{POSET}_δ . Here K_Z is defined by associating a δ_Z -prealgebraic lattice with the poset $k_Z(L)$ and a morphism $f : L \rightarrow M$ in $\delta_Z\mathbf{PALG}$ with the map $K_Z(f) : k_Z(L) \rightarrow k_Z(M)$ defined by

$$\forall x \in k_Z(L), K_Z(f)(x) = f(x);$$

Γ^Z is defined by assigning a poset P to the δ_Z -prealgebraic lattice $\Gamma^Z(L)$ and the σ^Z -continuous mapping $g : P \rightarrow Q$ to $\Gamma^Z(g) : \Gamma^Z(P) \rightarrow \Gamma^Z(Q)$ defined as follows:

$$\forall A \in \Gamma^Z(P), \Gamma^Z(g)(A) = cl_{\sigma^Z(Q)}(g(A)).$$

Then Γ^Z is left adjoint to K_Z with unit η_P and counit ϵ_P given by

$$\begin{aligned} \eta_P : P &\rightarrow K_Z\Gamma^Z(P) : p \mapsto \downarrow p, \forall p \in P, \text{ and} \\ \epsilon_L : \Gamma^Z K_Z(L) &\rightarrow L : E \mapsto \sup E, \forall E \in \Gamma^Z K_Z(L), \end{aligned}$$

respectively.

Proof. Step 1: Verify that functors K_Z and Γ^Z are well-defined by showing that $K_Z(f)$ and $\Gamma^Z(g)$ are morphisms in \mathbf{POSET}_δ , $\delta_Z\mathbf{PALG}$, respectively. We claim that $K_Z(f)$ is σ^Z -continuous, that is, $K_Z(f)(D^\delta \downarrow_{k_Z(L)}) \subseteq (K_Z(f)(D))^\delta \downarrow_{k_Z(M)}$ for any $D \in Z(k_Z(L))$. Since L is a complete lattice, by Lemma 5.8, $k_Z(L)$ is a *zcpo*. Thus we only need to prove that $K_Z(f)(\downarrow_{k_Z(L)} \sup_{k_Z(L)} D) \subseteq \downarrow_{k_Z(M)} \sup_{k_Z(M)} K_Z(f)(D)$. More precisely, to show $K_Z(f)(\downarrow \sup D \cap k_Z(L)) \subseteq \downarrow \sup K_Z(f)(D) \cap k_Z(M)$. From the fact that f has an upper adjoint, we know $f(\sup D) = \sup f(D)$ holds. So it is easy to see that

$$\begin{aligned} K_Z(f)(\downarrow \sup D \cap k_Z(L)) &\subseteq K_Z(f)(\downarrow \sup D) \cap k_Z(M) \\ &\subseteq \downarrow f(\sup D) \cap k_Z(M) \\ &= \downarrow \sup f(D) \cap k_Z(M) \\ &= \downarrow \sup K_Z(f)(D) \cap k_Z(M). \end{aligned}$$

Hence, $K_Z(f)$ is σ^Z -continuous.

We proceed to show $\Gamma^Z(g)$ has an upper adjoint and preserves the relation \prec_Z . It is obvious that $\Gamma^Z(g)$ preserves arbitrary sups in $\Gamma^Z(P)$, by Corollary O-3.5 in [6], $\Gamma^Z(g)$ has an upper adjoint. Moreover, the upper adjoint is given by

$$h : \Gamma^Z(Q) \rightarrow \Gamma^Z(P) : C \mapsto g^{-1}(C).$$

By Proposition 5.3, we know for any $\mathcal{C} \in \Gamma^Z(\Gamma^Z(Q))$, $\sup \mathcal{C} = \bigcup \mathcal{C}$. It follows that

$$h(\mathcal{C}^\delta) = g^{-1}(\downarrow \sup \mathcal{C}) = \downarrow \{g^{-1}(\bigcup \mathcal{C})\} = \downarrow \{\bigcup g^{-1}(C)\} = \downarrow \sup h(\mathcal{C}) = h(\mathcal{C})^\delta.$$

Therefore, from the conclusion of Lemma 5.7, we get that $\Gamma^Z(g)$ preserves \prec_Z . So $\Gamma^Z(g)$ is a morphism in $\delta_Z\mathbf{PALG}$.

Step 2: To show Γ^Z is left adjoint to K_Z in detail. Obviously, η_P is σ^Z -continuous, that is, a morphism in \mathbf{POSET}_δ . Now let L be a δ_Z -prealgebraic lattice and $f : P \rightarrow K_Z(L)$ σ^Z -continuous. We define

$$\bar{f} : \Gamma^Z(P) \rightarrow L : A \mapsto \sup f(A).$$

It is easy to find that $K_Z(\bar{f}) \circ \eta = f$. Thus for the remainder, we need to prove that \bar{f} is a unique morphism in $\delta_Z\mathbf{PALG}$ such that $K_Z(\bar{f}) \circ \eta = f$. Note that \bar{f} preserves arbitrary sups in $\Gamma^Z(P)$ and $\Gamma^Z(P)$ is a complete lattice, so \bar{f} has an upper adjoint, denoted by f^* . More specifically, for any $m \in L$,

$$\begin{aligned} f^*(m) &= \sup \bar{f}^{-1}(\downarrow m) \\ &= \sup \{C \in \Gamma^Z(P) : \bar{f}(C) \leq m\} \\ &= \sup \{\downarrow t \in \Gamma^Z(P) : \bar{f}(\downarrow t) \leq m\} \\ &= \sup \{\downarrow t \in \Gamma^Z(P) : f(t) \leq m\}. \end{aligned}$$

Then again by Lemma 5.7, we check that $f^*(B^\delta) \subseteq f^*(B)^\delta$ for any $B \in \Gamma^Z(L)$ to affirm \bar{f} preserves \prec_Z . Since L and $\Gamma^Z(P)$ are complete lattices, we only need to prove $f^*(\sup B) \leq \sup f^*(B)$. To this end, consider each $\downarrow x \in \Gamma^Z(P)$ which satisfies $f(x) \leq \sup B$, that is, $f(x) \in B^\delta$. Then $f(x) \in B$ as $f(x) \in k_Z(L)$. It follows that $\downarrow x \subseteq f^*(f(x))$, in addition, $\downarrow x \subseteq \sup f^*(B)$. Thus $f^*(\sup B) \leq \sup f^*(B)$ holds, hence, $f^*(B^\delta) \subseteq f^*(B)^\delta$. Besides, clearly, \bar{f} is unique. Therefore, we can conclude that Γ^Z is left adjoint to K_Z . \square

Next, we will give a monad on \mathbf{POSET}_δ . Before this, let us recall the following conclusion:

Proposition 5.11 [3] *Let $U : \mathcal{B} \rightarrow \mathcal{A}$ and $F : \mathcal{A} \rightarrow \mathcal{B}$ be functors such that F is left adjoint to U with $\eta : id \rightarrow UF$ and $\epsilon : FU \rightarrow id$ the unit and counit, respectively. Then $(UF, \eta, U\epsilon F)$ is a monad on \mathcal{A} .*

Now, by combining the above two conclusions, and $K_Z\Gamma^Z$ is written as δ , we obtain the following.

Theorem 5.12 *The endofunctor δ together with two natural transformation $\eta : id \rightarrow \delta$ and $\mu = \Gamma^Z\epsilon K_Z : \delta^2 \rightarrow \delta$ is a monad on the category \mathbf{POSET}_δ . More precisely, for each $P \in \mathbf{POSET}_\delta$, $\eta_P : P \rightarrow \delta(P)$ and $\mu_P : \delta^2(P) \rightarrow \delta(P)$ are defined as:*

$$\begin{aligned} \forall p \in P, \eta(p) &= \downarrow p, \\ \forall \mathcal{A} \in \delta^2(P), \mu(\mathcal{A}) &= \sup \mathcal{A}, \end{aligned}$$

respectively.

Recall that an *Eilenberg-Moore algebra* for a monad (T, η, μ) on a category \mathcal{C} is a pair (C, ξ) , where $\xi : TC \rightarrow C$ is a morphism in \mathcal{C} called a structure map which satisfies $\xi \circ \eta_C = id_C$ and $\xi \circ \mu_C = \xi T\xi$. In addition, we call a poset P δcpo if for any $A \in \delta(P)$, $\sup A$ exists.

Theorem 5.13 *There exists a structure map $\xi : \delta(P) \rightarrow P$ in \mathbf{POSET}_δ such that (P, ξ) is an Eilenberg-Moore algebra of the monad (δ, η, μ) if and only if P is a δcpo .*

Proof. (\Rightarrow): We claim that $\sup A = \xi(A)$ exists for any $A \in \delta(P)$. Consider each $a \in A$, from the facts that $\eta(a) = \downarrow a \subseteq A$ and ξ is order-preserving, we have $\xi(\eta(a)) \leq \xi(A)$. This implies $a \leq \xi(A)$ since $\xi \circ \eta_P = id_P$. Thus $\xi(A)$ is an upper bound of A . Assume that m is another upper bound of A , which means $A \subseteq \downarrow m = \eta(m)$. So $\xi(A) \leq \xi(\eta(m))$ by the monotonicity of ξ again. It follows that $\xi(A) \leq m$, and hence $\xi(A) = \sup A$.

(\Leftarrow): Since P is a δcpo , we can define $\xi : \delta(P) \rightarrow P$ by $C \mapsto \sup C$. One can easily verify that $\xi \circ \eta_P = id_P$ and $\xi \circ \mu_P = \xi \delta \xi$. For the remaining part, what we need to prove is that ξ is a morphism in \mathbf{POSET}_δ , that is, ξ is σ^Z -continuous. To this end, let \mathcal{A} be a Z -set of $\delta(P)$. Since for every upper bound y of $\xi(\mathcal{A})$, we have $\xi(A) = \sup A \leq y$ for each $A \in \mathcal{A}$, which implies $A \subseteq \downarrow y$ and so $\downarrow y$ is an upper bound of \mathcal{A} . Thus for every $B \in \mathcal{A}^\delta$, $B \subseteq \downarrow y$, which means $\xi(B) = \sup B \leq y$. It follows that $\xi(\mathcal{A}^\delta) = \{\xi(B) : B \in \mathcal{A}^\delta\} \subseteq \xi(\mathcal{A})^\delta$. Hence, ξ is σ^Z -continuous. \square

Refer to [11], if (T, η, μ) is a monad on the category \mathcal{C} , the category **T-ALG** consists of all T -algebras as objects and morphisms of T -algebras as morphisms. Here a morphism of T -algebras between (C, ξ) and (C', ξ') in **T-ALG** is a morphism $f : C \rightarrow C'$ in \mathcal{C} which satisfies $f \circ \xi = \xi' T f$.

Combining with the characterization of δ -algebras, we can deduce that $f : (P, \alpha) \rightarrow (Q, \beta)$ is a δ -algebra morphism if and only if $f(\sup A) = \sup f(A)$ for every $A \in \delta(P)$.

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