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Quantaloidal Completions of Order-enriched Categories and Their Applications

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Abstract

By introducing the concept of quantaloidal completions for an order-enriched category, relationships between the category of quantaloids and the category of order-enriched categories are studied. It is proved that quantaloidal completions for an order-enriched category can be fully characterized as compatible quotients of the power-set completion. As applications, we show that a special type of injective hull of an order-enriched category is the MacNeille completion; the free quantaloid over an order-enriched category is the Down-set completion.

Keywords: Quantaloid, order-enriched category, completion; injective hull, free quantaloid

1 Introduction

An order-enriched category is a locally small category such that the hom-sets are partially ordered sets and composition of morphisms preserve order in both variables. An order-enriched category with only one object can be viewed as a partially ordered semigroup. Thus order-enriched categories can be viewed as categorical generalization of partially ordered semigroups. Several works devoted to this subject are from computer science [15,33], especially with strong background of the study of programming languages. In 1979, M. Wand studied fixed-point constructions in order-enriched categories, which extended Scott's result based on continuous lattices. Note that an order-enriched category in the sense of [33] means a category with hom-sets not only ordered but also with certain completeness. Later, M. Smyth and G. Plotkin considered solving recursive domain equations in this framework [26]. In 2007, this ideal was further extended to the framework of bicategories [5]. In 1991, C. E. Martin, C. A. R. Hoare and He Jifeng studied pre-adjunctions in order enriched-categories [15]. In [15], the concepts of lax functors, natural transformations and pre-adjunctions are studied with the purpose to explain their understanding of programming languages. We also note that an order-enriched category in the sense of [15] means a

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category with hom-sets preordered. These works are all devoted to study special kind of order-enriched categories. There are little works devoted to study on them systematically.

A quantaloid $\mathcal{Q}$ [1, 20–22] is a category enriched in the symmetric monoidal closed category **Sup** of complete lattices and morphisms that preserve arbitrary sups. Just as every complete lattice is a special partially ordered set, every quantaloid is a special order-enriched categories. A quantaloid with only one object is a quantale [19], thus quantaloids are naturally viewed as quantales with many objects. Quantaloids were studied by Pitts [17] in investigating distributive categories of relations and topos theory under the name of sup-lattice enriched categories. In [1] quantaloids are studied in order to include a notion of type on the processes. Quantaloids and their applications were further developed in the monograph [22]. In recent years, Quantaloid-enriched categories received considerable attention [6,8,11,13,16,23–25,27–32].

The process of completion is a classic approach to study ordered structures. Various completion methods for ordered structures are developed with different characteristics [3, 4, 7, 9, 14, 18, 34]. Relationships between order-enriched categories and quantaloids have not received enough attention, though they have similar backgrounds and close relations. Inspired by research on completion methods for ordered semigroups and their applications [9, 12, 20, 34], this paper is devoted to study quantaloidal completions of order-enriched categories and their applications.

The contents of the paper are arranged as follows. Section 2 lists some preliminary notions and results about order-enriched categories and quantaloids. In Section 3, based on compatible nuclei on quantaloids, quantaloidal completions for an order-enriched category are fully characterized as compatible quotients of the power-set completion. In Section 4, two aspects of applications of quantaloidal completions are given. It is proved that the injective hull of an order-enriched category with respect to a special kind of morphisms is the MacNeille completion; the free quantaloid over an order-enriched category is the Down-set completion.

2 Preliminaries on order-enriched categories and quantaloids

For category theory, we refer to [2, 10]. Let $\mathcal{C}_0$ be the class of objects of a category $\mathcal{C}$. $\mathcal{C}(a,b)$ denotes the hom-set for $a,b \in \mathcal{C}_0$. For $a \in \mathcal{C}_0$, 1_a denotes the identity on a.

Definition 2.1 ([15]) An *order-enriched category* is a locally small category $\mathcal{A}$ such that:

(1) for $a,b \in \mathcal{A}_0$, the hom-set $\mathcal{A}(a,b)$ is a poset,

(2) composition of morphisms of $\mathcal{A}$ preserves order in both variables.

Definition 2.2 ([35]) Let $\mathcal{C}$, $\mathcal{D}$ be order-enriched categories. A *lax semifunctor* $F : \mathcal{C} \to \mathcal{D}$ is given by functions $F : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_{a,b} : \mathcal{C}(a,b) \to \mathcal{D}(Fa,Fb)$ for all $a,b \in \mathcal{C}_0$ such that $F_{a,b}$ is order-preserving and $(Fg) \circ (Ff) \leq F(g \circ f)$ for all $a,b,c \in \mathcal{C}_0$, $f \in \mathcal{C}(a,b), g \in \mathcal{C}(b,c)$. A *lax functor* $F : \mathcal{C} \to \mathcal{D}$ is a lax semifunctor such that $1_{Fa} \leq F(1_a)$ for all $a \in \mathcal{C}_0$. A *2-functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor such that

$$F_{a,b} : \mathcal{C}(a,b) \to \mathcal{D}(Fa,Fb)$$

is order-preserving for all $a,b \in \mathcal{C}_0$.

A *quantaloid* $\mathcal{Q}$ [22] is a category enriched in the symmetric monoidal closed category **Sup** of complete lattices and morphisms that preserve arbitrary sups. In elementary terms:

Definition 2.3 ([20]) A *quantaloid* is a locally small category $\mathcal{Q}$ such that:

(1) for $a,b \in \mathcal{Q}_0$, the hom-set $\mathcal{Q}(a,b)$ is a complete lattice,

(2) composition of morphisms of $\mathcal{Q}$ presevers sups in both variables.

In this paper, $\mathcal{Q}$ always denotes a small quantaloid, and $\mathcal{Q}_0$ denotes the set of its objects. The identity $\mathcal{Q}$-arrow on $q \in \mathcal{Q}_0$ will be denoted by 1_q. The greatest element of the complete lattice $\mathcal{Q}(p,q)$ will be denoted by $\top_{p,q}$. For a $\mathcal{Q}$-arrow $u : p \longrightarrow q$, we denote the domain and the codomain of u by $\mathrm{dom}(u)$ and $\mathrm{cod}(u)$, respectively. Given $\mathcal{Q}$-arrows $u : p \longrightarrow q$, $v : q \longrightarrow r$, the corresponding adjoints induced by the compositions $-\circ u : \mathcal{Q}(q,r) \longrightarrow \mathcal{Q}(p,r)$ and $v \circ - : \mathcal{Q}(p,q) \longrightarrow \mathcal{Q}(p,r)$ are denoted by $u \to_l -$ and $v \to_r -$ respectively.

For more details on quantaloids, we refer to [20, 22].

Definition 2.4 ([20]) Let $\mathcal{Q}$, $\mathcal{S}$ be quantaloids. A *quantaloidal homomorphism* $F : \mathcal{Q} \to \mathcal{S}$ is a functor such that

$$F : \mathcal{Q}(X,Y) \to \mathcal{S}(FX,FY)$$

is sup-preserving for all $X, Y \in \mathcal{Q}_0$.

A *quantaloidal isomorphism* is a quantaloidal homomorphism such that it is bijictive on objects and hom-sets.

Example 2.5 Let $\mathcal{A}$ be an order-enriched category.

(1) $\mathcal{P}(\mathcal{A})$ is a quantaloid [20]. The objects of $\mathcal{P}(\mathcal{A})$ are those of $\mathcal{A}$. For $a, b \in \mathcal{A}$, the hom-set $\mathcal{P}(\mathcal{A})(a,b)$=$\mathcal{P}(\mathcal{A}(a,b))$, the power set of the hom-set $\mathcal{A}(a,b)$. For $S \in \mathcal{P}(\mathcal{A})(a,b), T \in \mathcal{P}(\mathcal{A})(b,c)$, $T \circ S = \{g \circ f \mid g \in T, f \in S\}$.

(2) $\mathcal{D}(\mathcal{A})$ is a quantaloid. The objects of $\mathcal{D}(\mathcal{A})$ are those of $\mathcal{A}$. For $a, b \in \mathcal{A}$, the hom-set $\mathcal{D}(\mathcal{A})(a,b)$=$\mathcal{D}(\mathcal{A}(a,b))$, the set of down sets [4] of the hom-set $\mathcal{A}(a,b)$. For $S \in \mathcal{D}(\mathcal{A})(a,b), T \in \mathcal{D}(\mathcal{A})(b,c)$, $T \circ S = \downarrow \{g \circ f \mid g \in T, f \in S\}$. We note that $\downarrow 1_a \in \mathcal{D}(\mathcal{A}(a,a))$ is the identity morphism.

Definition 2.6 ([20]) Let $\mathcal{Q}$ be a quantaloid. A *quantaloidal nucleus* is a lax functor $j : \mathcal{Q} \to \mathcal{Q}$, which is the identity on the objects of $\mathcal{Q}$ and such that the maps $j_{a,b} : \mathcal{Q}(a,b) \to \mathcal{Q}(a,b)$ satisfy:

(1) $f \leq j_{a,b}(f)$ for all $f \in \mathcal{Q}(a,b)$,

(2) $j_{a,b}(j_{a,b}(f)) = j_{a,b}(f)$ for all $f \in \mathcal{Q}(a,b)$,

(3) $j_{b,c}(g) \circ j_{a,b}(f) \leq j_{a,c}(g \circ f)$ for all $g \in \mathcal{Q}(b,c)$ $f \in \mathcal{Q}(a,b)$.

For a quantaloidal nucleus j on a quantaloid $\mathcal{Q}$, let $\mathcal{Q}_j$ be the bicategory with the same objects as $\mathcal{Q}$ and $\mathcal{Q}_j(a,b) = \{f \in \mathcal{Q}(a,b) \mid j_{a,b}(f) = f\}$ for $a, b \in (\mathcal{Q}_j)_0$. Composition in $\mathcal{Q}_j$ is defined as follows: $g \circ_j f = j_{a,c}(g \circ f)$ for $f \in \mathcal{Q}_j(a,b)$, $g \in \mathcal{Q}_j(b,c)$.

Proposition 2.7 *([20]) If j is a quantaloidal nucleus on a quantaloid $\mathcal{Q}$, then $\mathcal{Q}_j$ is a quantaloid and $j : \mathcal{Q} \to \mathcal{Q}_j$ is a quantaloidal homomorphism.*

Proposition 2.8 *([20]) Let $\mathcal{S}$ be a subcategory of a quantaloid $\mathcal{Q}$, which contains all the objects of $\mathcal{Q}$. Then, $\mathcal{S}$ is a quotient quantaloid of the form $\mathcal{Q}_j$ for some quantaloidal nucleus j iff*

(1) each hom-set $\mathcal{S}(a,b)$ is closed under infs, and

(2) if $f \in \mathcal{S}(a,c)$, then $g \to_l f \in \mathcal{S}(b,c)$ for all $g \in \mathcal{Q}(a,b)$ and $h \to_r g \in \mathcal{S}(a,b)$ for all $h \in \mathcal{Q}(b,c)$.

3 Quantaloidal completions of order-enriched categories

In order to study quantaloidal completions of order-enriched categories, let us begin with the concept of a compatible nucleus on a quantaloid.

Definition 3.1 Let $\mathcal{A}$ be an order-enriched category, $j : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ a quantaloidal nucleus. j is said to be *compatible* if for $a, b \in \mathcal{A}_0$, $f \in \mathcal{A}(a,b)$, we have $j_{a,b}(\{f\}) = \downarrow f$.

Definition 3.2 Let $\mathcal{A}$ be an order-enriched category, $\mathcal{Q}$ a quantaloid, $F : \mathcal{A} \to \mathcal{Q}$ a 2-functor. The pair $(F, \mathcal{Q})$ is said to be a *quantaloidal completion* of $\mathcal{A}$, if the following conditions are satisfied:

(1) $F : \mathcal{A}_0 \to \mathcal{Q}_0$ is bijective,

(2) $F_{a,b} : \mathcal{A}(a,b) \to \mathcal{Q}(Fa,Fb)$ is an order embedding for all $a, b \in \mathcal{A}_0$,

[4] A set D in a poset P is a down set, if $D = \downarrow D$, where $\downarrow D = \{x \mid \exists d \in D, \text{ s. t. } x \leq d\}$.

(3) for every $a, b \in \mathcal{A}_0$ and $f \in \mathcal{Q}(Fa, Fb)$, there exists $U_f \subseteq \mathcal{A}(a, b)$ such that $f = \bigvee F(U_f)$.

Theorem 3.3 *If j is a compatible nucleus on an order-enriched category $\mathcal{A}$, then $(F_j, \mathcal{P}(\mathcal{A})_j)$ is a quantaloidal completion of $\mathcal{A}$, where $F_j : \mathcal{A} \to \mathcal{P}(\mathcal{A})_j$ is defined as follows:*

(1) $F_j : \mathcal{A}_0 \to (\mathcal{P}(\mathcal{A})_j)_0$ is the identity map,

(2) $F_j(f) = \downarrow f$ for every $f \in \mathcal{A}(a, b)$, $a, b \in \mathcal{A}_0$.

Proof. By definition, $F_j : \mathcal{A}_0 \to (\mathcal{P}(\mathcal{A})_j)_0$ is bijective, and $F_j : \mathcal{A}(a, b) \to \mathcal{P}(\mathcal{A})_j(a, b)$ is an order-embedding. For $S \in \mathcal{P}(\mathcal{A})_j(a, b)$, we have $S = j_{a,b}(S) = j_{a,b}(\bigcup_{f\in S}\{f\}) = \bigvee_{f\in S}^{\mathcal{P}(\mathcal{A})_j(a,b)} j_{a,b}(\{f\}) = \bigvee_{f\in S}^{\mathcal{P}(\mathcal{A})_j(a,b)} \downarrow f = \bigvee_{f\in S}^{\mathcal{P}(\mathcal{A})_j(a,b)} F_j(f)$. This completes the proof. □

Corresponding to several classical completion methods of posets and ordered semigroups, we can obtain a series of compatible nucleus. We leave detail to the reader.

Example 3.4 (Down-set completion) Let $\mathcal{A}$ be an order-enriched category. Define a lax functor $\downarrow: \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ as follows:

(1) $\downarrow: \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0$ is the identity map,

(2) $\downarrow_{a,b}(S) = \downarrow S$ for $S \in \mathcal{P}(\mathcal{A})(a, b)$, $a, b \in \mathcal{P}(\mathcal{A})_0$.

Then $\downarrow$ is a compatible nucleus. The quotient corresponding to $\downarrow$ is $\mathcal{D}(\mathcal{A})$.

Example 3.5 (MacNeille completion) Let $\mathcal{A}$ be an order-enriched category. Define a lax functor $\mathrm{cl} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ as follows:

(1) $\mathrm{cl} : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0$ is the identity map,

(2) $\mathrm{cl}_{a,b}(S) = \{f \in \mathcal{P}(\mathcal{A})(a, b) \mid \forall g \in \mathcal{P}(\mathcal{A})(a', a), h \in \mathcal{P}(\mathcal{A})(b, b'), k \in \mathcal{P}(\mathcal{A})(a, b), h \circ S \circ g \subseteq \downarrow k \text{ implies } h \circ f \circ g \leq k\}$ for $S \in \mathcal{P}(\mathcal{A})(a, b)$, $a, b \in \mathcal{P}(\mathcal{A})_0$.

Then cl is a compatible nucleus.

Example 3.6 (Equivariant completion) Let $\mathcal{A}$ be an order-enriched category. Suppose $S \subseteq \mathcal{P}(\mathcal{A})(a, b)$. If the join of S exists and is preserved by composition, i.e., $f \circ (\bigvee S) = \bigvee(f \circ S)$, $(\bigvee S) \circ g = \bigvee(S \circ g)$ whenever the composition is well-defined, then $\bigvee S$ is said to be an *equivariant join* with respect to S. Clearly, every $f \in \mathcal{P}(\mathcal{A})(a, b)$ is an equivariant join respect to $\downarrow f$. If k is an equivariant join with respect to S, then $g \circ k$ (resp., $k \circ h$) is an equivariant join with respect to $g \circ S$ (resp., $S \circ h$), whenever the composition is well-defined. For $S \subseteq \mathcal{P}(\mathcal{A})(a, b)$, let

$$S^{EJ} = \{f \in \mathcal{P}(\mathcal{A})(a, b) \mid \exists T \subseteq S, \text{s.t. } f = \bigvee T \text{ is an equivariant join with respect to } T\}.$$

Let $EJ(\mathcal{A})$ be the subcategory of $\mathcal{A}$, which contains all the objects of $\mathcal{A}$. The hom-sets

$$EJ(\mathcal{A})(a, b) = \{S \in \mathcal{D}(\mathcal{A})(a, b) \mid S = S^{EJ}\}.$$

Then $EJ(\mathcal{A})$ is a quotient of $\mathcal{A}$ such that $\downarrow f \in EJ(\mathcal{A})(a, b)$ for every $f \in \mathcal{P}(\mathcal{A})(a, b)$. Consequently, the corresponding quantaloidal nucleus is compatible.

For an order-enriched category $\mathcal{A}$, $CN(\mathcal{A})$ denotes the class of all compatible nuclei on $\mathcal{P}(\mathcal{A})$, $QC(\mathcal{A})$ denotes the set of all quantaloidal completions of $\mathcal{A}$.

Let $\mathcal{A}$ be an order-enriched category, $(F, \mathcal{Q}) \in QC(\mathcal{A})$. Define $j_{(F,\mathcal{Q})} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ as follows:

(1) $j_{(F,\mathcal{Q})} : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0$ is the identity map,

(2) $j_{(F,\mathcal{Q})}(S) = \{f \in \mathcal{A}(a, b) \mid F(f) \leq \bigvee_{g\in S} F(g)\}$ for every $S \in \mathcal{P}(\mathcal{A})(a, b), a, b \in \mathcal{A}_0$.

Lemma 3.7 *Let $\mathcal{A}$ be an order-enriched category, $(F, \mathcal{Q}) \in QC(\mathcal{A})$. Then $j_{(F,\mathcal{Q})}$ is a compatible nucleus on $\mathcal{P}(\mathcal{A})$.*

Proof. By definition, $j_{(F,\mathcal{Q})} : \mathcal{P}(\mathcal{A})_0 \to \mathcal{P}(\mathcal{A})_0$ is bijective, $j_{(F,\mathcal{Q})} : \mathcal{P}(\mathcal{A})(a,b) \to \mathcal{P}(\mathcal{A})(a,b)$ is order preserving and increasing for all $a, b \in \mathcal{A}_0$. Suppose $S \in \mathcal{P}(\mathcal{A})(a,b), f \in j_{(F,\mathcal{Q})}(j_{(F,\mathcal{Q})}(S))$. Then, $F(f) \leq \bigvee_{g \in j_{(F,\mathcal{Q})}(S)} F(g)$. For every $g \in j_{(F,\mathcal{Q})}(S)$, we have $F(g) \leq \bigvee_{k \in S} F(k)$. Thus, $F(f) \leq \bigvee_{k \in S} F(k)$. Consequently, $f \in j_{(F,\mathcal{Q})}(S)$. So we can conclude that $j_{(F,\mathcal{Q})} \circ j_{(F,\mathcal{Q})} = j_{(F,\mathcal{Q})}$. Thus, $j_{(F,\mathcal{Q})} : \mathcal{P}(\mathcal{A})(a,b) \to \mathcal{P}(\mathcal{A})(a,b)$ is a closure operator for every $a, b \in \mathcal{A}_0$.

Suppose $K \in \mathcal{P}(\mathcal{A})(b,c)$, $S \in \mathcal{P}(\mathcal{A})(a,b)$. Then $j_{(F,\mathcal{Q})}(K) \circ j_{(F,\mathcal{Q})}(S) = \{g \circ f \mid g \in \mathcal{A}(b,c), f \in \mathcal{A}(a,b), F(g) \leq \bigvee_{k \in K} F(k), F(f) \leq \bigvee_{t \in S} F(t)\}$. If $F(g) \leq \bigvee_{k \in K} F(k), F(f) \leq \bigvee_{t \in S} F(t)$, then $F(g \circ f) \leq \bigvee_{k \in K, t \in S} F(k) \circ F(t) = \bigvee_{k \in K, t \in S} F(k \circ t) \leq \bigvee_{p \in K \circ S} F(p)$. Thus, $j_{(F,\mathcal{Q})}(K) \circ j_{(F,\mathcal{Q})}(S) \subseteq j_{(F,\mathcal{Q})}(K \circ S)$.

For $f_0 \in \mathcal{A}(a,b)$, by the fact that $F : \mathcal{A}(a,b) \to \mathcal{Q}(F(a), F(b))$ is an order embedding, we have $j_{(F,\mathcal{Q})}(\{f_0\}) = \{f \in \mathcal{A}(a,b) \mid F(f) \leq F(f_0)\} = \downarrow f_0$.

So we can conclude that $j_{(F,\mathcal{Q})}$ is a compatible nucleus on $\mathcal{P}(\mathcal{A})$. □

Theorem 3.8 *Let $\mathcal{A}$ be an order-enriched category, $(F, \mathcal{Q}) \in QC(\mathcal{A})$. Then $\mathcal{Q}$ is quantaloidal isomorphism to $\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}$.*

Proof. Let $F^{-1} : \mathcal{Q}_0 \to \mathcal{A}_0$ be the inverse of the map $F : \mathcal{A}_0 \to \mathcal{Q}_0$. Define $G : \mathcal{Q} \to \mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}$ as follows:

(1) $G(a) = F^{-1}(a)$ for every $a \in \mathcal{Q}_0$,

(2) $G(p) = \{f \in \mathcal{A}(F^{-1}(c), F^{-1}(d)) \mid F(f) \leq p\}$ for every $p \in \mathcal{Q}(c,d)$.

Then $G : \mathcal{Q}_0 \to (\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}})_0$ is bijective. For $f \in j_{(F,\mathcal{Q})}(G(p))$, we have $F(f) \leq \bigvee_{g \in G(p)} F(g) \leq p$, thus $f \in G(p)$. Thus, $j_{(F,\mathcal{Q})}(G(p)) \subseteq G(p)$. Consequently, $G(p) = j_{(F,\mathcal{Q})}(G(p)) \in \mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}$. Thus, G is well-defined.

Suppose $a, b \in \mathcal{Q}_0, S \subseteq \mathcal{Q}(a,b)$. Then $G(\bigvee S) = \{f \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid F(f) \leq \bigvee S\}$,

$$\begin{aligned}
\bigvee_{t \in S}^{\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}} G(t) &= j_{(F,\mathcal{Q})}\left(\bigcup_{t \in S} G(t)\right) \\
&= j_{(F,\mathcal{Q})}\left(\bigcup_{t \in S} \{g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid F(g) \leq t\}\right) \\
&= j_{(F,\mathcal{Q})}\{g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid \exists t \in S, \text{ s.t. } F(g) \leq t\} \\
&= \{f \in \mathcal{A}(F^{-1}(a), F^{-1}(b)) \mid F(f) \leq \bigvee \{F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), \exists t \in S, \\
&\qquad \text{s.t. } F(g) \leq t\}\}.
\end{aligned}$$

For $s_0 \in S$, we have

$$\begin{aligned}
s_0 &= \bigvee \{F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), F(g) \leq s_0\} \\
&\leq \bigvee \{F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), \exists t \in S, \text{ s.t. } F(g) \leq t\}.
\end{aligned}$$

Thus, $\bigvee S \leq \bigvee \{F(g) \mid g \in \mathcal{A}(F^{-1}(a), F^{-1}(b)), \exists t \in S, \text{ s.t. } F(g) \leq t\}$, whence $G(\bigvee S) \leq \bigvee_{t \in S}^{\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}} G(t)$. The inverse inequality holds trivially. Therefore, $G(\bigvee S) = \bigvee_{t \in S}^{\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}} G(t)$.

For $a \in \mathcal{Q}_0$, we have $G(1_a) = \{f \in \mathcal{A}(F^{-1}(a), F^{-1}(a)) \mid F(f) \leq 1_a\} = \downarrow 1_{G(a)}$, which is the identity in $\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}$.

Suppose $f \in \mathcal{Q}(a,b), g \in \mathcal{Q}(b,c)$. Then

$$\begin{aligned}
G(g) \circ_{j_{(F,\mathcal{Q})}} G(f) &= j_{(F,\mathcal{Q})}(G(g) \circ G(f)) \\
&= \{t \in \mathcal{A}(F^{-1}(a), F^{-1}(c)) \mid F(t) \leq \bigvee \{F(h) \mid h \in G(g) \circ G(f)\}\}.
\end{aligned}$$

Since,

$$
\begin{aligned}
&\bigvee\{F(h) \mid h \in G(g)\circ G(f)\}\\
=&\bigvee\{F(t_2\circ t_1) \mid t_1 \in \mathcal{A}(F^{-1}(a),F^{-1}(b)), t_2 \in \mathcal{A}(F^{-1}(b),F^{-1}(c)), F(t_1)\le f, F(t_2)\le g\}\\
=&\left(\bigvee\{F(t_2) \mid t_2 \in \mathcal{A}(F^{-1}(b),F^{-1}(c)), F(t_2)\le g\}\right)\circ\left(\bigvee\{F(t_1) \mid t_1 \in \mathcal{A}(F^{-1}(a),F^{-1}(b)), F(t_1)\le f\}\right)\\
=&g\circ f,
\end{aligned}
$$

we have $G(g)\circ_{j_{(F,\mathcal{Q})}} G(f) = \{t \in \mathcal{A}(F^{-1}(a),F^{-1}(c)) \mid F(t)\le g\circ f\} = G(g\circ f)$.

So we can conclude that G is a quantaloidal homomorphism.

Suppose $p_1, p_2 \in \mathcal{Q}(c,d)$ with $G(p_1) = G(p_2)$. Then $p_1 = \bigvee F(G(p_1)) = \bigvee F(G(p_2)) = p_2$. Thus, $G : \mathcal{Q}(c,d) \to (\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}})(F^{-1}(c),F^{-1}(d))$ is injective for all $c,d \in \mathcal{Q}_0$.

Suppose $S \in \mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}(a,b)$. Then $S \subseteq \mathcal{A}(a,b)$. For every $f \in \mathcal{A}(a,b)$, we have $G(F(f)) = \{g \in \mathcal{A}(F^{-1}(a),F^{-1}(b)) \mid F(g) \le F(f)\} = \{g \in \mathcal{A}(F^{-1}(a),F^{-1}(b)) \mid g \le f\} =\downarrow f$. Thus, $S = j_{(F,\mathcal{Q})}(S) = j_{(F,\mathcal{Q})}\left(\bigcup_{f\in S}\{f\}\right) = \bigvee_{f\in S}^{\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}(a,b)} j_{(F,\mathcal{Q})}(\{f\}) = \bigvee_{f\in S}^{\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}(a,b)} \downarrow f = \bigvee_{f\in S}^{\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}}(a,b)} G(F(f)) = G\left(\bigvee_{f\in S}^{\mathcal{Q}(F(a),F(b))} F(f)\right)$. Thus, $G : \mathcal{Q}(a,b) \to (\mathcal{P}(\mathcal{A})_{j_{(F,\mathcal{Q})}})(F^{-1}(a),F^{-1}(b))$ is surjective. Therefore, G is a quantaloidal isomorphism. □

As a combination of the above results, we obtain that quantaloidal completions of an order-enriched category $\mathcal{A}$ are completely determined by compatible quantaloidal nuclei on $\mathcal{P}(\mathcal{A})$.

Theorem 3.9 *Let $\mathcal{A}$ be an order-enriched category. Then $(F,\mathcal{Q})$ is a quantaloidal completion of $\mathcal{A}$ if and only if there is a compatible nucleus j on $\mathcal{P}(\mathcal{A})$ such that $\mathcal{Q}$ is quantaloidal isomorphism to $\mathcal{P}(\mathcal{A})_j$.*

4 Applications

In this section, we shall give two kinds of applications for the quantaloidal completions of order-enriched categories.

4.1 Injective constructs of order-enriched categories

Let $\mathbf{O\text{-}Cat}_l$ be the category of order-enriched categories and lax semifunctors. Let $\mathcal{E}^{ls}_{\le}$ be the class of all lax semifunctors in $\mathbf{O\text{-}Cat}_l$ satisfying the following conditions:

(1) $F : \mathcal{C}_0 \to \mathcal{D}_0$ is bijective;

(2) $F(f_1)\circ F(f_2)\circ\cdots\circ F(f_n) \le F(f)$ implies $f_1\circ f_2\circ\cdots\circ f_n \le f$ for $f_1\circ f_2\circ\cdots\circ f_n, f \in C(a,b)$, $a,b \in \mathcal{C}_0$.

Lemma 4.1 *In the category $\mathbf{O}$-$\mathbf{Cat}_l$, every retract of a quantaloid is a quantaloid.*

Proof. Let $\mathcal{S}$ be a retract of a quantaloid $\mathcal{Q}$. Then there exist lax semifunctors $I : \mathcal{S} \to \mathcal{Q}$ and $F : \mathcal{Q} \to \mathcal{S}$ such that $F\circ I = \mathrm{id}_{\mathcal{S}}$. Suppose $S, T \in \mathcal{S}_0$. Then $\mathcal{S}(X,T)$ is a retract of $\mathcal{Q}(IX,IY)$. By the fact that $\mathcal{Q}(IX,IY)$ is a complete lattice, we can deduce that $\mathcal{S}(X,Y)$ is a complete lattice and $F(\bigvee I(A))$ is the least upper bound of A in $\mathcal{S}(X,Y)$. Suppose $A \subseteq \mathcal{S}(X,Y)$, $g \in \mathcal{S}(Y,Y')$, $t \in \mathcal{S}(X',X)$. Then $g\circ(\bigvee A)$ is an upper bound of $g\circ A$. If h is an upper bound of $g\circ A$, then $I(g)\circ\bigvee_{f\in A} I(f) = \bigvee_{f\in A}(I(g)\circ I(f)) \le \bigvee_{f\in A}(I(g\circ f)) \le I(h)$. Thus, $h = FI(h) \ge F\left(I(g)\circ\bigvee_{f\in A} I(f)\right) \ge FI(g)\circ F\left(\bigvee_{f\in A} I(f)\right) = g\circ(\bigvee A)$. Thus, $g\circ(\bigvee A) = \bigvee(g\circ A)$. Similarly, we have $(\bigvee A)\circ t = \bigvee(A\circ t)$. Therefore, $\mathcal{S}$ is a quantaloid. □

Theorem 4.2 *Let $\mathcal{A}$ be an order-enriched category. Then $\mathcal{A}$ is $\mathcal{E}^{ls}_{\le}$-injective in $\mathbf{O}$-$\mathbf{Cat}_l$ if and only if $\mathcal{A}$ is a quantaloid.*

Proof. Suppose $\mathcal{Q}$ is a quantaloid, $H : \mathcal{S} \to \mathcal{T}$ a morphism in $\mathcal{E}^{ls}_{\leq}$, and $F : \mathcal{S} \to \mathcal{Q}$ a morphism in $\mathbf{O\text{-}Cat}_l$. Define $G : \mathcal{T} \to \mathcal{Q}$ as follows:

(1) $GX = FH^{-1}(X)$, $\forall X \in \mathcal{T}_0$;

(2) $G(g) = \bigvee\{F(f_1) \circ F(f_2) \circ \cdots \circ F(f_n) \mid H(f_1) \circ H(f_2) \circ \cdots \circ H(f_n) \leq g, f_1 \circ f_2 \circ \cdots \circ f_n \in \mathcal{S}(H^{-1}X, H^{-1}Y)\}$ for $g \in \mathcal{T}(X,Y)$, $X, Y \in \mathcal{T}_0$.

Then $G : \mathcal{T}(X,Y) \to \mathcal{Q}(GX, GY)$ is order-preserving for $X, Y \in \mathcal{T}_0$. Suppose $g_1 \in \mathcal{T}(X,Y)$, $g_2 \in \mathcal{T}(Y,Z)$. Since composition in a quantaloid distribute over arbitrary joins, we can deduce that $G(g_2) \circ G(g_1) \leq G(g_2 \circ g_1)$. Thus $G : \mathcal{T} \to \mathcal{Q}$ is a lax semifunctor. For $X \in \mathcal{S}$, we have $GH(X) = FH^{-1}H(X) = F(X)$. For $f \in \mathcal{S}(X,Y)$, we have $H(f) \in \mathcal{T}(HX, HY)$. By the fact that F is a morphism in $\mathbf{O\text{-}Cat}_l$, we can deduce that $GH(f) = \bigvee\{F(f_1) \circ F(f_2) \circ \cdots \circ F(f_n) \mid H(f_1) \circ H(f_2) \circ \cdots \circ H(f_n) \leq H(f), f_1 \circ f_2 \circ \cdots \circ f_n \in \mathcal{S}(X,Y)\} = F(f)$. Thus, $G : \mathcal{T} \to \mathcal{Q}$ is a morphism in $\mathbf{O\text{-}Cat}_l$ such that $GH = F$. So we can conclude that $\mathcal{Q}$ is $\mathcal{E}^{ls}_{\leq}$-injective.

Conversely, suppose $\mathcal{A}$ is $\mathcal{E}^{ls}_{\leq}$-injective in $\mathbf{O\text{-}Cat}_l$. Define $F : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ as follows:

(1) $F : \mathcal{A}_0 \to \mathcal{D}(\mathcal{A})_0$ is the identity map;

(2) $F(f) = \downarrow f$ for $f \in \mathcal{A}(a,b)$, $a, b \in \mathcal{A}_0$.

Then its routine to check that $F \in \mathcal{E}^{ls}_{\leq}$. Thus, for the identity functor $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, there is a lax semifunctor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$ such that $GF = \mathrm{id}_{\mathcal{A}}$. So, $\mathcal{A}$ is a quantaloid, as it is a retract of the quantaloid $\mathcal{D}(\mathcal{A})$. □

Let $\mathcal{A}$ be an order-enriched category. Define $\eta : \mathcal{A} \to \mathcal{P}(\mathcal{A})_{\mathrm{cl}}$ as follows:

(1) $\eta : \mathcal{A}_0 \to (\mathcal{P}(\mathcal{A})_{\mathrm{cl}})_0$ is the identity map;

(2) $\eta(f) = \downarrow f$ for $f \in \mathcal{A}(a,b)$, $a, b \in \mathcal{A}_0$.

Then it is routine to check that η is a lax semifunctor and it is $\mathcal{E}^{ls}_{\leq}$-essential. As the proof is quite similar to that of Theorem 5.8 in [12], we leave it to the reader.

Theorem 4.3 *Let $\mathcal{A}$ be an order-enriched category. Then $\mathcal{P}(\mathcal{A})_{\mathrm{cl}}$ is an $\mathcal{E}^{ls}_{\leq}$-injective hull of $\mathcal{A}$ in* $\mathbf{O\text{-}Cat}_l$.

4.2 Free quantaloids

Let **LocSm** be the category of locally small categories and functors between them. Let **O-Cat** be the category of order-enriched categories and 2-functors. Let **Qtlds** be the category of quantaloids and quantaloidal homomorphisms.

Theorem 4.4 *The functor $\mathcal{D}$:* **O-Cat** $\to$ **Qtlds** *is left adjoint to the forgetful functor* **Qtlds** $\to$ **O-Cat**.

Proof. Let $\mathcal{A}$ be an order-enriched category. Define $\eta : \mathcal{A} \to \mathcal{D}(\mathcal{A})$ as follows:

(1) $\eta : \mathcal{A}_0 \to (\mathcal{D}(\mathcal{A}))_0$ is the identity map;

(2) $\eta(f) = \downarrow f$ for $f \in \mathcal{A}(a,b)$, $a, b \in \mathcal{A}_0$.

Then η is a 2-functor in **O-Cat**.

Suppose that $\mathcal{Q}$ is a quantaloid and that $F : \mathcal{A} \to \mathcal{Q}$ is a 2-functor in **O-Cat**. Define $\bar{F} : \mathcal{D}(\mathcal{A}) \to \mathcal{Q}$ as follows:

(1) $\bar{F}(a) = F(a)$ for every $a \in (\mathcal{D}(\mathcal{A}))_0$;

(2) $\bar{F}(S) = \bigvee\{F(f) \mid f \in S\}$ for every $S \in \mathcal{D}(\mathcal{A})(a,b)$.

For $a \in (\mathcal{D}(\mathcal{A}))_0$, we have $\bar{F}(\downarrow 1_a) = \bigvee\{F(f) \mid f \in \downarrow 1_a\} = \bigvee\{F(f) \mid f \leq 1_a\} = F(1_a) = 1_{F(a)}$. For $T \in \mathcal{D}(\mathcal{A})(b,c), S \in \mathcal{D}(\mathcal{A})(a,b)$, we have $\bar{F}(T \circ S) = \bigvee\{F(h) \mid h \in T \circ S\} = \bigvee\{F(g \circ f) \mid g \in T, f \in S\} = \bigvee\{F(g) \mid g \in T\} \circ \bigvee\{F(f) \mid f \in S\} = \bar{F}(T) \circ \bar{F}(S)$. For $S_i \in \mathcal{D}(\mathcal{A})(a,b), i \in I$, we have $\bar{F}\left(\bigvee_{i \in I} S_i\right) = \bar{F}\left(\bigcup_{i \in I} S_i\right) = \bigvee\{F(f) \mid f \in \bigcup_{i \in I} S_i\} = \bigvee_{i \in I} \bigvee\{F(f) \mid f \in S_i\} = \bigvee_{i \in I} \bar{F}(S_i)$. Thus, $\bar{F}$ is a quantaloidal homomorphism. Furthermore, we can check that $\bar{F} \circ \eta = F$.

Suppose $G : \mathcal{D}(\mathcal{A} \to \mathcal{Q}$ is a quantaloidal homomorphism with $G \circ \eta = F$. Then we have

(1) $\forall a \in (\mathcal{D}(\mathcal{A}))_0$, $\bar{F}(a) = \bar{F}(\eta(a)) = F(a) = (G \circ \eta)(a) = G(a)$;

(2) $\forall S \in \mathcal{D}(\mathcal{A})(a,b)$, $\bar{F}(S) = \bar{F}(\bigcup\{\downarrow f \mid f \in S\}) = \bigvee_{f\in S} \bar{F}(\downarrow f) = \bigvee_{f\in S} \bar{F}(\eta(f)) = \bigvee_{f\in S} F(f) = \bigvee_{f\in S}(G\circ\eta)(f) = \bigvee_{f\in S} G(\downarrow f) = G\left(\bigvee_{f\in S}\downarrow f\right) = G(S)$. Thus, $\bar{F} : \mathcal{D}(\mathcal{A} \to \mathcal{Q}$ is the unique quantaloidal homomorphism such that $\bar{F}\circ\eta = F$. □

Every locally small category can be viewed as an order-enriched category with the discrete order on hom-sets. We know $\mathcal{D}(\mathcal{A}) = \mathcal{P}(\mathcal{A})$ for every locally small category with discrete order on hom-sets. Thus, we can recover the following results [20].

Corollary 4.5 *The functor* $\mathcal{P}$: **LocSm** $\to$ **Qtlds** *is left adjoint to the forgetful functor* **Qtlds** $\to$ **LocSm**.

5 Conclusion and some further work

In this paper, we only considered quantaloidal completions for order-enriched categories. As order-enriched category with other completeness have deep applications in domain theory [15, 26, 33], other types of completions and applications deserve to be developed further.

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